1.1 Heuristic derivations are a good start.

Quite often the right answer is first found by guessing, or at best, shaky logic. In a sense, this process gives us the proverbial “answer in the back of the book.” Getting the right answer by such means is not sleazy — it’s an important and natural part of research. The danger of heuristics is that many people never go back and clean up their analyses. Once your heuristic analysis gives you an “answer in the back of the book,” it becomes much easier to go back and do the analysis correctly because you now know where you are headed and what you expect to find. If your rigorous analysis does not lead to the same result as your heuristic analysis, you must determine what component of your thinking was wrong.

1.2 Logical statements

Below, “A” and “B” denote statements (such as “the solution is unique” or “today is Tuesday” or “the moon is made of green cheese”). The statements may or may not be true. Whenever someone makes a logical statement, you must “turn that statement around” and look at it from different perspectives. Making such critical evaluations will often help you better understand the statement if it is true, or it will help you see a way to disprove the statement if it is false.

Below, we present the most common kinds of logical statements encountered in applied mathematics. We will show how each statement may be viewed in different ways. In the literature, statements are often presented in a lengthy sentence form. To evaluate the statement, you might want to convert it to symbolic form, manipulate the symbolic form, and transform back to a sentence that is equivalent to the original statement but phrased in a different way.

Logical statements may be interpreted in tabular form by using “truth tables.” Truth tables use binary arithmetic in which a statement is zero if it is false and unity if it is true. When two statements, A and B, are “multiplied” the result equals zero if either A or B is false; the product equals one only if A and B are both true. Thus, in binary arithmetic, the product $AB$ represents the statement “both A and B are true.” The sum $A+B$ is assigned the value of zero only if A and B are both zero; otherwise, the sum $A+B$ is assigned the value 1. Thus, $A+B$ represents the statement “A or B (or both) are true.

Logical statements may also be interpreted in graphical form through the use of Venn diagrams in which a statement is regarded as a set and is illustrated as a closed area in the Venn diagram. The empty set $\emptyset$ represents “false” and the universal symbol $\Omega$ represents truth. Shaded regions in a Venn diagram graphically illustrate scenarios. If the interior of the region representing A is shaded, then the shaded region represents all scenarios in which A is true. The mathematical statement $A = \emptyset$ would be taken to mean that $A$ is false. Complicated scenarios involving two statements, A and B, may also be illustrated in this manner. As with the truth
tables, the world is divided into four possible situations depending on whether $A$ and $B$ are true or false. Graphically these are the four regions in the Venn diagram defined by the areas representing $A$ and $B$. The scenarios in which “$A$ or $B$ (or both) are true” would be described in set notation as $A \cup B$ and would be illustrated by shading the interiors of both $A$ and $B$. The scenarios in which “$A$ and $B$ are both true” would be described in set notation as $A \cap B$ and would be illustrated by shading only that region that is inside both $A$ and $B$. The bottom four Venn diagrams in the figure illustrate the four possible scenarios based on whether $A$ and $B$ are true or false; these four regions correspond to the four cells in a truth table.

Ordinary sentences and their symbolic, tabular, and graphical representations are all useful tools for evaluating logical statements. When you encounter a new theorem, you will be doing yourself a great service by first identifying the basic nature of the statement (if-then, existence, definition, etc.) and then recasting and reinterpreting it in all of these logical representations available to you. Spending just a few minutes doing this will help ingrain the theorem and its meaning in your mind.

**If-then statements**

An “if-then” statement merely indicates that one thing follows from another. Importantly, these statements carry no indication of whether the converse relationship is true. All of the following are different ways of saying exactly the same thing:

1. $\text{If } A \text{ then } B$.
2. $A \text{ implies } B$.
3. $A$ is sufficient $B$ for $B$ true.
4. B is necessary for A to be true.
5. B is true if A is true.
6. A ⇒ B
7. A ⊆ B (i.e., A is contained in or is identical to B)
8. A ∩ B = ∅ (i.e., there is no instance of A that is not also B)
9. A ∪ B = Ω (i.e., statement B is true and/or A is false)

Graphically, this statement may be displayed in a Venn diagram or truth table as shown at right. Note that if-then statements do not provide complete information about the truth table. One might be tempted to place a “1” in the lower right corner of the truth table, but this is not allowed because we don’t know whether it is even possible for A to be true. We only know that if A is true, then so is B. Therefore we only know it’s impossible for A to be true and B to be simultaneously false. This allows us to identify the lower left corner of the truth table as an impossible scenario.

Converses

The converse of “A ⇒ B” is “B ⇒ A”. The converse of an if-then statement is not necessarily true! Compare the term “converse” with the term “contrapositive” defined later.

One-way relationships

Suppose it is known that one thing implies the other, but not vice-versa. In other words, a statement is known to be true while the converse of the statement is known to be false. Then it is often helpful to emphasize the one-way nature of the relationship. The following are equivalent:

1. If A then B but not vice-versa.
2. A ⇒ B, but the converse of B is not true.
3. A ⇒ B but A ≠ B .
4. A ⇒ B
5. A is sufficient but not necessary for B to be true.
6. B is necessary but not sufficient for A to be true.

If-and-only-if statements

An “if-and-only-if” statement says that one thing follows from the other and vice versa. In other words, both the statement and its converse are true. All of the following are different ways of saying exactly the same thing:

1. If A then B and vice-versa.
2. A implies B and B implies A.
3. A is necessary and sufficient for B to be true.
4. B is true if and only if A is true.
5. B is equivalent to or tantamount to A.
6. B if A.
Incidentally, definitions are always if-only-if statements. If \( A \) is the statement used to define the term and if \( A \iff B \), then \( B \) may be used as an alternative definition of the term.

**Complement or negation**

An exclamation point preceding a statement or bar over a statement negates the statement. The following are equivalent:

1. \( A \) is not true.
2. \( \neg A \)
3. \( \overline{A} \)
4. \( A! \)

**Contrapositive**

The contrapositive of the statement \( A \Rightarrow B \) is \( \overline{B} \Rightarrow \overline{A} \). Note the distinction between the converse and the contrapositive. If it is known that \( A \) is true then the contrapositive is also true (whereas the converse may or may not be true).

**Proof by contradiction, contrapositive, and by counterexample**

An extremely important logic theorem says that a relationship of the form “\( A \) implies \( B \)” is true if and only if “\( \neg B \) implies \( \neg A \)”, and it follows that.

1. If \( A \) implies \( B \), then \( \neg B \) implies \( \neg A \), and vice versa.
2. \( A \Rightarrow B \) is equivalent \( \overline{B} \Rightarrow \overline{A} \) \( \quad \)
3. \( A \Rightarrow B \) is equivalent \( \overline{B} \Rightarrow \overline{A} \) \( \quad \)
4. \( A \iff B \) is equivalent \( \overline{B} \Rightarrow \overline{A} \) \( \quad \)

Given that \( A \Rightarrow B \), the equivalent statement \( \overline{B} \Rightarrow \overline{A} \) is called the “contrapositive” of the original relationship. Note the important difference between the equivalent contrapositive form (which is always true) and the converse (which may or may not be true.)

A clever analysis technique often called “proof by contradiction” proves a statement of the form \( A \Rightarrow B \) by instead proving the contrapositive. For this method, \( \neg A \) is assumed that \( A \) is false and it is then shown that \( B \) must be false. This is commonly interpreted as a “contradiction” to the original premise \( A \). More correctly, proof by contradiction merely proves the contrapositive \( \overline{B} \Rightarrow \overline{A} \), and the above theorem is then invoked to conclude that \( A \Rightarrow B \).

**Existence**

There are two ways to prove the existence of something. You can actually produce the desired object (showing that it satisfies all required conditions), or you can indirectly prove existence. For example, you can prove that there exists a solution to the equation \( x^3 + x - 2 = 0 \) by actually producing a solution \( (x = 1) \). Alternatively, an indirect proof would follow from noting that the function \( f(x) = x^3 + x - 2 \) is negative at \( x = 0 \) and positive at \( x = 2 \) and, being continuous, the function must therefore cross the x-axis somewhere between 0 and 2. Typically, your audience will be most satisfied if you actually produce the thing that you claim exists. However, sometimes it’s not possible to actually produce the desired item. Also, even if you can pro-
duce the actual item, it is occasionally not desirable to present it; an example would be an analytical expression that takes five pages to write down.

**Proof by induction**

Proof by induction is commonly used to prove statements that hold for an arbitrary integer $n$. For example, if $q$ is the interest rate for a one-time investment $I_0$, then the value of the investment after $n$ compounding periods is $I_0(1 + q)^n$. To properly prove such a statement by induction, one must (1) prove it is true for $n = 1$, (2) Assume that it is true for $n$ and then prove that it must be true for $n+1$.

“**For all**” $\forall$

The following are equivalent statements:

1. If $A$ is true for all $a$, then $B$
2. $(A \forall a) \Rightarrow B$
3. If $B$ is false, then there exists at least one $a$ such that $A$ is false.
4. $\exists a \in A$

When something is true “for all” $x$, then it must be true for particular $x$’s. You can often explore some well selected $x$’s to draw conclusions. Trivial example: if $pq=0$ for all $q$, then $pq=0$ for any particular choices of $q$. If we choose $q=1$ and hence $p$ must be zero. Note the distinction between the statement “$pq=0$ for all $q$” and the statement “$pq=0.$” Without the added requirement that the statement holds for all $q$ then we could only conclude that $p$ and/or $q$ must be zero. In continuum mechanics, a tensor is called “isotropic” if its components are the same in all coordinate systems. To determine the form of the most general isotropic tensor, it is useful to first impose the definition for just 90 degree rotations; any resulting conditions become necessary for isotropy and imposing these conditions can simplify the general derivation of sufficient conditions.

“**For each**” $\forall$

When something is true “for each” $x$, then the details of the statement may vary for particular $x$’s. For example, for each real symmetric tensor, there exists an orthogonal principal basis in which the components form a diagonal matrix — the basis is generally different for each different tensor.

**1.2.1 Units (dimension)**

Once you have completed a derivation — heuristic or rigorous — one of the most useful ways of checking the validity of your analysis is to ensure that it has the right units. Check the physical dimensions of each step in your solution. Each term in a sum must have the same units. The units of the final result must have the units you would expect physically. Unit checking is a strong reason for you to maintain a “nomenclature” table that includes the physical dimensions of physical quantities.
1.2.2 Special cases

When you derive a result in symbols (rather than numbers), you can often substitute values for the symbols for which you know the answer. For example, if you derive a result for a transversely isotropic material, you should check that it gives the correct answer in the limit of isotropy. If you derive a result for the motion of a body in the presence of wind resistance, then you should check that it reduces to the appropriate classical form when you set the wind resistance to zero.

1.2.3 Symmetries and cyclic properties

Results in symbols (rather than numbers) should exhibit appropriate symmetries. For example, the formula for the volume of a box having sides of length a, b, and c should yield the same value for a box having sides of length c, b, a.

1.2.4 Personalize

Relate the problem to something familiar (like money).

Doubt any solution that required skills in which you are personally weak. For example, if your solution required long division (and you are notoriously bad at it), check it again.

1.2.5 Notation

Always use impeccable notation. Always check that the two rules for Einstein’s summation convention are satisfied. Always check that the tensor order of every term in an expression is the same (e.g., you can’t add a vector plus a tensor).

1.2.6 Analogy

If the problem is analogous to some other problem, expect (1) the methods of solution to be similar, (2) the simpler result to be needed for the more complicated problem, and/or (3) the final solutions to be similar in form. Examples: compatibility is like solvability of dyad equation, derivative of the inverse of a tensor is like derivative of $1/x$, integral of $n_{nn}$ depends on the analogously derived intermediate result for the integral of $nn$).

1.2.7 Sign and Order of magnitude

The result should be “physically reasonable.” Suppose, for example, you derive the deflection of a 3cm beam due to a 1 gram weight; if your answer comes out to be negative 53 meters, you should recognize that something is seriously wrong — both the sign and the magnitude are ludicrous! Suppose you are in a closed book exam and you can’t recall whether the sine of $60^\circ$ equal $\sqrt{3}/2$ or $2/\sqrt{3}$, then you should pick the one that is less than one!
1.2.8 Experience, intuition, and “tricks”

patterns

Given an integral involving a square root, expect the answer to involve the same square root or else an arcsin or an arcsec. If you don’t get one, you better know why.

Given a sum of fractions, you cannot get rid of a denominator that appears only once.

Given a polynomial (on an exam or homework), one of the roots is probably a factor of the constant term.

Mnemonics

A mnemonic is any device that helps you remember something. For example, the word “criteria” is plural. It is grammatically wrong to say “the criteria is.” One should instead say “the criterion is” and “the criteria are.” A silly mnemonic to help you remember this is that “the more there are, the fewer letters you should use!” This mnemonic applies to numerous latin words (e.g., phenomena/phenomenon).

In curvilinear coordinates, there are two ways to place an index on a vector component. If the index is a subscript (as in $v_1$), then the component is called “covariant.” If the index is a superscript (as in $v^1$), then the component is called “contravariant.” It’s easy to get these terms mixed up unless one knows the cute mnemonic that “co-go-below.”

Quite often, a good mnemonic might simply be a fast way to view the structure of something. Consider using integration by parts to integrate $xe^x$. Unsophisticated students would write down the following tedious procedure:

\[
\begin{align*}
\text{let } u &= x \text{ and } dv = e^{3x}. \\
\text{Then } du &= 1 \text{ and } v = \frac{1}{3}e^{3x}.
\end{align*}
\]

Integration by parts gives

\[
\int u dv = uv - \int v du = \frac{x}{3}e^{3x} - \int \frac{1}{3}e^{3x} dx = \frac{x}{3}e^{3x} - \frac{1}{9}e^{3x}
\]

Showing this level of detail is appropriate for undergraduate textbooks, but not for advanced analysis. Furthermore, even for your own analyses, this amount of writing and thinking is unnecessary if one revisits integration by parts. Namely integration by parts is a procedure you use when integrating the product of two things. One of the two factors must simplify when differentiated and the other must at least not become more complicated when integrated. To write down integration by parts without having to introduce $u$ and $v$ substitutions, you should change the way that you remember integration by parts. Simply say to yourself the phrase “integrate*same—minus—integrate*differentiate.” The term “integrate” means to write down the integral of the factor to be integrated. The terms “same” means to write down the other factor unchanged. The second term will involve both the integral of the easily integrated factor and the derivative of the easily differentiated factor. Doing integration by parts in this way forces you to remember the formula by its structure which also lends insight into the technique.

In continuum mechanics, the only effective way to remember and interpret formulas is to memorize the structure of the placement of indices. Don’t memorize formulas — memorize the basic structure of formulas. This skill is by far one of the most difficult for new students to embrace.
Special theorems
Suppose you are solving a polynomial for a physical quantity that you know must be positive. Then you might be able to use Descarte’s rule of signs to argue that there is only one solution. This is much faster (and therefore less error prone) than actually solving for all the roots.

Quite often, recasting new theorems into their alternative forms discussed earlier can give you an arsenal of special theorems for elegant solutions.

1.2.9 Computer checks
Symbolic
Mathematica, Matlab, Maple, IDL etc. There is really no excuse for making mistakes when multiplying matrices or performing other such simple numerical operations. There’s no excuse for getting an integral wrong. If it’s truly important for you to get the answer right, then you can. There are numerous symbolic math programs available to help with such mundane tasks.

Numerical
Suppose you’re doing an algebraic integral and you’re not sure if you carried out the algebra correctly. Then just assign some numbers to the constants in the integral and solve the integral on your scientific calculator to see if the numerical solution matches what you get when you substitute the same constants into your analytical solution.

Suppose you are simplifying some function of \( x \). Once you are done, substitute some arbitrary value for \( x \) into the starting and ending expressions. If you don’t get the same result, there must be a mistake somewhere. Best way to find the mistake? Substitute your test value for \( x \) into all the intermediate expressions until you find the step where the answer deviates.

Suppose you’re in an exam and can’t recall whether the hyperbolic cosine is \( \frac{1}{2}(e^x + e^{-x}) \) or \( \frac{1}{2}(e^x - e^{-x}) \). Just use your calculator to compute, say, cosh(0). When you see that the answer is 1, then you’ll know that the right definition of cosh must have the “+” sign.

1.2.10 Counterexamples
Once you start to believe that your answer is correct, the last step is to aggressively attempt to disprove it. After all, that’s precisely what your reviewers (or colleagues) will try to do. Remember, disproving things is much easier than proving them — one need only find a single example that disproves the claim.

1.2.11 Double Check
Once you have found what you consider to be the correct answer, always substitute it back into the original problem to verify that it satisfies all restrictions of the problem statement.
1.3 Notation of set theory and formal analysis

1.3.1 Mathematical symbols

∃ “such that”
∃ “there exists” (¢ does not exist)
∃! “there exists a unique”
-><- “contradiction”
A ⇒ B “A implies B”
A⇒B “A implies B, but B does not imply A”
A⇐B “B implies A”
A⇔B “A implies B and vice versa”
y=x “y is a function of x”
A= {a: propositions involving a} “Definition of a set A”
∀ “for all”
∈ “for each”
α “is proportional to”
∴ “therefore”

1.3.2 Sets

A∩B “Intersection between A and B”
A⊂B “A is contained in B” (¢ not contained)
A⊃B “B is contained in A”
A “The set of all reals”
A∪B “A union B”
x∈A “x is a member of the set A”
x∉A “x is not a member of A”

1.3.3 Tensors

or “second-order inner product”
⊗ “fourth-order inner product”
⊕ “vector addition”
⊗ “dyadic multiplication”
— “full product”

u “(right-operating) gradient of u”
u “(right-operating) gradient of u”