BONUS PROBLEM WORTH 15 HOMEWORK POINTS

In homework #8, there is a formula for the gradient of the dot product between two vectors:

\[ \vec{\nabla} (\vec{v} \cdot \vec{w}) = (\vec{\nabla} \vec{v}) \cdot \vec{w} + (\vec{\nabla} \vec{w}) \cdot \vec{v} \quad (1) \]

However, if you look up the gradient of a dot product in a math handbook such as the CRC, you will find the following *horrendous* formula:

\[ \vec{\nabla} (\vec{v} \cdot \vec{w}) = (\vec{w} \cdot \vec{\nabla}) \vec{v} + (\vec{v} \cdot \vec{\nabla}) \vec{w} + \vec{w} \times (\vec{\nabla} \times \vec{v}) + \vec{v} \times (\vec{\nabla} \times \vec{w}), \quad (2) \]

In eq (2), the notation \((\vec{w} \cdot \vec{\nabla}) \vec{v}\) means the same thing as \(\vec{w} \cdot (\vec{\nabla} \vec{v})\). In other words, the \(i\)th component of the first term on the right-hand-side of eq (2) is \(w_k \frac{\partial v_i}{\partial x_k} = w_k v_{i,k}\).

This is not the same as of the first term on the right-hand-side of eq (1), for which the \(i\)th component is \(\left(\frac{\partial}{\partial x_i} v_k\right) w_k = w_k v_{k,i}\).

(a) Use indicial notation to prove that Eqs. (1) and (2) are both correct. Which of these equations would you rather use in your applications?

(b) In this class, our proofs of formulas like this have employed indicial notation with respect to a rectangular CARTESIAN basis. The last step in our indicial derivations is to express the final result in direct notation. Explain why this process yields a direct notation formula that is valid for ANY basis (even nonorthonormal curvilinear).

(c) Consider cylindrical coordinates. Then, for example, \(\vec{v} = v_r \vec{e}_r + v_\theta \vec{e}_\theta + v_z \vec{e}_z\), where the components are all functions of \((r, \theta, z)\).

(i) GRADS: Explain why \(\vec{v} \cdot \vec{w} = v_r w_r + v_\theta w_\theta + v_z w_z\), despite the fact that this is a curvilinear coordinate system.

(ii) EVERYONE: Because the coordinates are curvilinear, the formulas for the gradient take on a different form. The radial component of the first term on the right hand side of Eq. (1) is \((\vec{\nabla} \vec{v})_{rr} w_r + (\vec{\nabla} \vec{v})_{r\theta} w_\theta + (\vec{\nabla} \vec{v})_{rz} w_z\), where the \(rr, r\theta, rz\) components of \(\vec{\nabla} \vec{v}\) are obtained from the formulas provided in the book for the gradient of vector in cylindrical coordinates [Beware: when the book says \(\vec{\nabla} \vec{v}\), it means what we call \(\vec{\nabla} \vec{v} = (\vec{\nabla} \vec{v})^T\). Thus, you must use the transpose of the book’s formulas if you really want \(\vec{\nabla} \vec{v}\)]. Using the book’s formulas, write out an explicit expression for the radial component of the first term on the right hand side of Eq. (1) if \(v_r = r\theta, v_\theta = \theta r^2, v_z = r^4 \sin \theta, w_r = r, w_\theta = z\theta, \) and \(w_z = 0\).