Laplace Transforms

Why use Laplace transforms?
1) Provide an algebraic solution to (linear) differential equations.
2) Can be used for frequency domain analysis of a system.
3) Eliminates the need to solve the convolution integral.

Consider a time function, \( f(t) \), that is well-behaved such that \( f(t)e^{-\sigma t} \) is absolutely integrable if \( \sigma \) is sufficiently large. The Fourier transform is:

\[
F\{e^{-\sigma t}f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-(\sigma + j\omega)t} \, dt = F\left(\frac{\sigma + j\omega}{j}\right) = F_1(\sigma + j\omega)
\]

the inverse is:

\[
f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(\sigma + j\omega)e^{(\sigma + j\omega)t} \, d\omega
\]

substitute: \( s = \sigma + j\omega \):

\[
F_1(s) = \int_{-\infty}^{\infty} f(t)e^{-st} \, dt
\]

and:

\[
f(t) = \frac{1}{2\pi}\int_{\sigma - j\infty}^{\sigma + j\infty} F_a(s)e^{st} \, ds
\]

This is the two-sided or bilateral Laplace transform and the inverse. Usually in engineering problems, the functions are limited for only positive time. Singularity functions are an example of these. This results in the one-sided or unilateral transform.

\[
F(s) = \int_{-\infty}^{\infty} f(t)u(t)e^{-st} \, dt = \int_{0}^{\infty} f(t)e^{-st} \, dt
\]

Finding the Laplace transform
1) Look up in tables
2) Solve the transform integral
Laplace Transform Theorems

1) Linearity

\[ L\{af_1(t) + bf_2(t)\} = aF_1(s) + bF_2(s) \]

where \( a \) and \( b \) are constants

2) Scaling

\[ L\{f(at)\} = \frac{1}{a}F(s/a) \]

3) Delay

\[ L\{f(t - t_0)u(t - t_0)\} = F(s)e^{-t_0s} \]

4) Differentiation of time functions

\[ L\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0) \]

5) Integration

\[ L\left\{\int_0^t f(\lambda)d\lambda\right\} = \frac{1}{s}F(s) \]

6) s-shift

\[ F(s + a) = L\{e^{-at}f(t)u(t)\} \]

7) Convolution

\[ L\{f_1(t)*f_2(t)\} = F_1(s)F_2(s) \]

8) Product

\[ L\{f_1(t)f_2(t)\} = F_1(s)*F_2(s) \]

9) Initial value theorem

\[ \lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s) \]

10) Final value theorem

\[ \lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s) \]
Inverse Laplace Transforms

Techniques

1) Look up in tables

2) If not in the tables, then break the function up into parts that are in the tables
   a) Partial fractions
   b) Method of Residues

3) Solve the inverse transform integral