Problem 2.53

The relationship between the nonlinear spring's displacement, \( x_s(t) \) and its force, \( f_s(t) \) is

\[
x_s(t) = 1 - e^{-f_s(t)}
\]

Solving for the force,

\[
f_s(t) = -\ln(1 - x_s(t))
\]

Writing the differential equation for the system by summing forces,

\[
\frac{d^2 x(t)}{dt^2} + \frac{dx(t)}{dt} - \ln(1 - x(t)) = f(t)
\]

Letting \( x(t) = x_o + \delta x \) and \( f(t) = 1 + \delta f \), linearize \( \ln(1 - x(t)) \).

\[
\ln(1 - x) - \ln(1 - x_o) = \frac{d\ln(1 - x)}{dx} \bigg|_{x=x_o} \delta x
\]

Solving for \( \ln(1 - x) \),

\[
\ln(1 - x) = \ln(1 - x_o) - \frac{1}{1 - x_o} \delta x = \ln(1 - x_o) - \frac{1}{1 - x_o} \delta x
\]

When \( f = 1 \), \( \delta x = 0 \). Thus from Eq. (1), \( 1 = -\ln(1 - x_o) \). Solving for \( x_o \),

\[
1 - x_o = e^{-1}, \text{ or } x_o = 0.6321.
\]

Substituting \( x_o = 0.6321 \) into Eq. (3),

\[
\ln(1 - x) = \ln(1 - 0.6321) - \frac{1}{1 - 0.6321} \delta x = -1 - 2.718 \delta x
\]

Placing this value into Eq. (2) along with \( x(t) = x_o + \delta x \) and \( f(t) = 1 + \delta f \), yields the linearized differential equation,

\[
\frac{d^2 \delta x}{dt^2} + \frac{d\delta x}{dt} + 1 + 2.718 \delta x = 1 + \delta f
\]

or

\[
\frac{d^2 \delta x}{dt^2} + \frac{d\delta x}{dt} + 2.718 \delta x = \delta f
\]

Taking the Laplace transform and rearranging yields the transfer function,

\[
\frac{\delta x(s)}{\delta f(s)} = \frac{1}{s^2 + s + 2.718}
\]
Problem 4.2 & 4.3

2.

a. \( C(s) = \frac{5}{s(s+5)} = \frac{1}{s} - \frac{1}{s+5} \). Therefore, \( c(t) = 1 - e^{-5t} \).

Also, \( T = \frac{1}{5} \), \( T_r = \frac{2.2}{a} = \frac{2.2}{5} = 0.44 \), \( T_s = \frac{4}{a} = \frac{4}{5} = 0.8 \).

b. \( C(s) = \frac{20}{s(s+20)} = \frac{1}{s} - \frac{1}{s+20} \). Therefore, \( c(t) = 1 - e^{-20t} \). Also, \( T = \frac{1}{20} \), 

\( T_r = \frac{2.2}{a} = \frac{2.2}{20} = 0.11 \), \( T_s = \frac{4}{a} = \frac{4}{20} = 0.2 \).

3.

Program:

'(a)'
num=5;
den=[1 5];
Ga=tf(num,den)
subplot(1,2,1)
step(Ga)
title(''(a)'')
'(b)'
num=20;
den=[1 20];
Gb=tf(num,den)
subplot(1,2,2)
step(Gb)
title(''(b)'')

Computer response:

ans =

(a)

Transfer function:

\[ \frac{5}{s + 5} \]

ans =

(b)

Transfer function:

\[ \frac{20}{s + 20} \]
Problem 4.8

8.

a. Pole: -2; \( c(t) = A + Be^{-2t} \); first-order response.

b. Poles: -3, -6; \( c(t) = A + Be^{-3t} + Ce^{-6t} \); overdamped response.

c. Poles: -10, -20; Zero: -7; \( c(t) = A + Be^{-10t} + Ce^{-20t} \); overdamped response.

d. Poles: (-3+j3\sqrt{15}), (-3-j3\sqrt{15}); \( c(t) = A + Be^{-3t} \cos(3\sqrt{15}t + \phi) \); underdamped.

e. Poles: j3, -j3; Zero: -2; \( c(t) = A + B \cos(3t + \phi) \); undamped.

f. Poles: -10, -10; Zero: -5; \( c(t) = A + Be^{-10t} + Cte^{-10t} \); critically damped.

Problem 4.16

16. 
\[ \%OS = e^{-\xi \pi} / \sqrt{1 - \xi^2} \times 100. \]
Dividing by 100 and taking the natural log of both sides,
\[ \ln \left( \frac{\%OS}{100} \right) = -\frac{\xi \pi}{\sqrt{1 - \xi^2}}. \]
Squaring both sides and solving for \( \xi^2 \), \( \xi^2 = \frac{\ln^2 \left( \frac{\%OS}{100} \right)}{\pi^2 + \ln^2 \left( \frac{\%OS}{100} \right)}. \)
Taking the negative square root, \( \xi = \frac{-\ln \left( \frac{\%OS}{100} \right)}{\sqrt{\pi^2 + \ln^2 \left( \frac{\%OS}{100} \right)}}. \)
Problem 4.20

20.

a. \( \omega_n^2 = 121 \text{ rad/s}^2, 2\zeta\omega_n = 13.2 \). Therefore \( \zeta = 0.6, \omega_n = 11. \)
\[
T_s = \frac{4}{\zeta \omega_n} = 0.606 \text{ s}; \quad T_P = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}}
\]
\[= 0.357 \text{ s}; \quad \% OS = e^{-\zeta \pi \sqrt{1 - \zeta^2}} \times 100 = 9.48 \%; \quad \omega_n T_r = (1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1); \]

therefore, \( T_r = 0.168 \text{ s} \).

b. \( \omega_n^2 = 0.04 \text{ rad/s}^2, 2\zeta\omega_n = 0.02 \). Therefore \( \zeta = 0.05, \omega_n = 0.2. \)
\[
T_s = \frac{4}{\zeta \omega_n} = 400 \text{ s}; \quad T_P = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}} = 15.73 \text{ s}; \quad \% OS = e^{-\zeta \pi \sqrt{1 - \zeta^2}} \times 100 = 85.45 \%; \quad \omega_n T_r = (1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1); \]

therefore, \( T_r = 5.26 \text{ s} \).

c. \( \omega_n^2 = 1.05 \times 10^7 \text{ rad/s}^2, 2\zeta\omega_n = 1.6 \times 10^3 \). Therefore \( \zeta = 0.247, \omega_n = 3240. \)
\[
T_s = \frac{4}{\zeta \omega_n} = 0.005 \text{ s}; \quad T_P = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}} = 0.001 \text{ s}; \quad \% OS = e^{-\zeta \pi \sqrt{1 - \zeta^2}} \times 100 = 44.92 \%; \quad \omega_n T_r = (1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1); \]

therefore, \( T_r = 3.88 \times 10^{-4} \text{ s} \).

Problem 4.25

25.

a. Writing the equation of motion yields, \( (3s^2 + 15s + 33)X(s) = F(s) \)

Solving for the transfer function,
\[
\frac{X(s)}{F(s)} = \frac{1/3}{s^2 + 5s + 11}
\]

b. \( \omega_n^2 = 11 \text{ rad/s}^2, 2\zeta\omega_n = 5. \) Therefore \( \zeta = 0.754, \omega_n = 3.32. \)
\[
T_s = \frac{4}{\zeta \omega_n} = 1.6 \text{ s}; \quad T_P = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}}
\]
\[= 1.44 \text{ s}; \quad \% OS = e^{-\zeta \pi \sqrt{1 - \zeta^2}} \times 100 = 2.7 \%; \quad \omega_n T_r = (1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1); \]

therefore, \( T_r = 0.69 \text{ s} \).
Problem 5.3

3. Split $G_3$ and combine with $G_2$ and $G_4$. Also use feedback formula on $G_6$ loop.

Push $G_2 + G_3$ to the left past the pickoff point.

Using the feedback formula and combining parallel blocks,

Multiplying the blocks of the forward path and applying the feedback formula,

Problem 5.7

7. Push $G_2$ to the left past the pickoff point and push $G_3$ to the left past the pickoff point.

Combine parallel $G_3$ and unity.

Push $G_3+1$ to the left past the summing junction.

Thus,

Applying the feedback formula,

$$T(s) = \frac{\frac{G_4(G_2G_3 + G_2(G_3 + 1))}{1 - G_4G_2G_3(G_3 + 1) + G_4(G_1G_3 + G_2(G_3 + 1))}}{\frac{G_4(G_2G_3 + G_2(G_3 + 1))}{1 - G_4G_2G_3(G_3 + 1) + G_4(G_1G_3 + G_2(G_3 + 1))}}$$
Problem 5.11

11. \[ T(s) = \frac{225}{s^2 + 12s + 225} \]. Therefore, \( 2\zeta \omega_n = 12 \), and \( \omega_n = 15 \). Hence, \( \zeta = 0.4 \).

\[ \% OS = e^{\frac{-\zeta \pi}{\sqrt{1-\zeta^2}}} \times 100 = 16.3\% \; ; \; T_s = \frac{4}{\zeta \omega_n} = 0.667 \; ; \; T_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}} = 0.229. \]

Problem 5.26, 5.27 & 5.28

26.

\[ \Delta = 1 + [G_2G_3G_4 + G_3G_4 + G_4 + 1] + [G_3G_4 + G_4]; T_1 = G_1G_2G_3G_4; \Delta_1 = 1. \] Therefore,

\[ T(s) = \frac{T_1 \Delta_1}{\Delta} = \frac{G_1G_2G_3G_4}{2 + G_2G_3G_4 + 2G_3G_4 + 2G_4} \]

27.
Closed-loop gains: \( G_2G_4G_6G_7H_3; G_2G_5G_6G_7H_3; G_3G_4G_6G_7H_3; G_3G_5G_6G_7H_3; G_6H_1; G_7H_2 \)

Forward-path gains: \( T_1 = G_1G_2G_4G_6G_7; \; T_2 = G_1G_2G_5G_6G_7; \; T_3 = G_1G_3G_4G_6G_7; \; T_4 = G_1G_3G_5G_6G_7 \)

Nontouching loops 2 at a time: \( G_6H_1G_7H_2 \)

\[ \Delta = 1 - [H_3G_6G_7(G_2G_4 + G_2G_5 + G_3G_4 + G_3G_5) + G_6H_1 + G_7H_2] + [G_6H_1G_7H_2] \]

\( \Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = 1 \)

\[ T(s) = \frac{T_1 \Delta_1 + T_2 \Delta_2 + T_3 \Delta_3 + T_4 \Delta_4}{\Delta} = \frac{G_1G_2G_4G_6G_7 + G_1G_2G_5G_6G_7 + G_1G_3G_4G_6G_7 + G_1G_3G_5G_6G_7}{1 - H_3G_6G_7(G_2G_4 + G_2G_5 + G_3G_4 + G_3G_5) + G_6H_1 + G_7H_2 + G_6H_1G_7H_2} \]

28.
Closed-loop gains: \( -s^2; -\frac{1}{s}; -\frac{1}{s}; -s^2 \)

Forward-path gains: \( T_1 = s; \; T_2 = \frac{1}{s^2} \)

Nontouching loops: None

\( \Delta = 1 - (-s^2 - \frac{1}{s} - \frac{1}{s} - s^2) \)

\( \Delta_1 = \Delta_2 = 1 \)

\[ G(s) = \frac{T_1 \Delta_1 + T_2 \Delta_2}{\Delta} = \frac{s + \frac{1}{s^2}}{1 + (s^2 + \frac{1}{s} + \frac{1}{s} + s^2)} = \frac{s^3+1}{2s^4+s^2+2s} \]