Introduction to Compressible Flow

\[ \frac{D\rho}{Dt} \neq 0 \]

The density of a gas changes significantly along a streamline.

Compressible Flow

Definition of Compressibility: the fractional change in volume of the fluid element per unit change in pressure.

\[ \frac{dV}{V} = \frac{d\rho}{\rho} \]

Compressible Flow

1. Mach Number: \( M = \frac{V}{c} = \frac{\text{local velocity}}{\text{speed of sound}} \)

2. Compressibility becomes important for High Speed Flows where \( M > 0.3 \)
   - \( M < 0.3 \) – Subsonic & incompressible
   - \( 0.3 < M < 0.8 \) – Subsonic & compressible
   - \( 0.8 < M < 1.2 \) – Transonic flow – shock waves appear in mixed subsonic and sonic flow regime
   - \( 1.2 < M < 3.0 \) – Supersonic – shock waves are present but NO subsonic flow
   - \( M > 3.0 \) – Hypersonic Flow, shock waves and other flow changes are very strong
Compressible Flow

3. Significant changes in velocity and pressure result in density variations throughout a flow field.

4. Large Temperature variations result in density variations.

As a result we now have two new variables we must solve for: $T$ and $\rho$.

We need 2 new equations. We will solve: mass, linear momentum, energy and an equation of state.

Important Effects of Compressibility on Flow

1. **Choked Flow** – a flow rate in a duct is limited by the sonic condition.
2. **Sound Wave/Pressure Waves** – rise and fall of pressure during the passage of an acoustic/sound wave. The magnitude of the pressure change is very small.
3. **Shock Waves** – nearly discontinuous property changes in supersonic flow. (Explosions, high speed flight, gun firing, nuclear explosion)
4. A pressure ratio of 2:1 will cause sonic flow.

Applications

1. **Nozzles and Diffusers and converging diverging nozzles**
2. **Turbines, fans & pumps**
3. **Throttles – flow regulators, an obstruction in a duct that controls pressure drop.**
4. **One Dimensional Isentropic Flow – compressible pipe flow.**
Approach

- Control volume approach
- Steady, One-dimension, Uniform Flow
- Additional Thermodynamics Concepts are needed
- Restrict our analysis to ideal gases

Thermodynamics

- Equation of State – Ideal Gas Law
  \[ p = \rho RT \]
  \[ R = \frac{R_u}{M_w} \]  Universal Gas Constant
  \[ M_w = \text{Molecular mass of air} \]
  \[ R_u = 8314 \text{l/kmol \cdot K} \]
  \[ M_w = 28 \text{g/kmol} \]

Temperature is absolute and the specific volume is (volume per unit mass):
\[ v = \frac{1}{\rho} \]

Thermodynamics – Internal Energy & Enthalpy

- Internal Energy – individual particle kinetic energy.
  Summation of molecular vibrational and rotational energy.
  \[ \bar{u} = \bar{u}(v, T) \]
  \[ d\bar{u} = \left( \frac{\partial \bar{u}}{\partial T} \right)_v dT + \left( \frac{\partial \bar{u}}{\partial v} \right)_T dv \]

- For an ideal gas \( \bar{u} = \bar{u}(T) \)
  \[ d\bar{u} = c_v dT \]

- Recall from our integral form of the Energy Equation for Enthalpy of an ideal gas: \( h = \bar{u} + pv \)
  \[ h = h(T) \]
  \[ dh = c_v dT \]
Thermodynamics – Internal Energy & Enthalpy

\[ dh = d\bar{u} + RdT \]

Substituting:

\[ dh = c_p dT \quad d\bar{u} = c_v dT \]

\[ dh = d\bar{u} + RdT \]

\[ c_v dT = c_p dT + RdT \]

\[ c_p = c_v + R \]

\[ c_p - c_v = R = \text{const} \]

\[ \frac{\beta}{\rho} = -RT \]

Substituting:

\[ \frac{\beta}{\rho} = c_v \frac{dV}{dT} \]

Define the ratio of specific heats:

\[ k = \frac{c_p}{c_v} = \text{const} \]

Then,

\[ c_v = \frac{kR}{k-1} \]

\[ c_p = \frac{R}{k-1} \]

For Air:

\[ c_v = 1004 \text{ J/kg-K} \]

\[ k = 1.4 \]

The 2nd Law of Thermodynamics & Isentropic Processes

We define entropy by:

\[ ds = \left( \frac{\delta Q}{T} \right)_s \]

Combining the 1st and 2nd Laws gives us Gibb’s Equation

\[ Tds = dh - \frac{dp}{\rho} \]

\[ Tds = c_v dT - \frac{dp}{\rho} \quad dh = c_p dT \]

\[ \int ds = \int c_v \frac{dT}{T} - \frac{1}{\rho} \frac{dp}{p} \quad \int \frac{1}{\rho} \frac{dp}{p} = \frac{R}{p} \]
### The 2nd Law of Thermodynamics & Isentropic Processes

\[ s_2 - s_1 = \frac{c_p}{\rho} \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1} \]

For an isentropic process: adiabatic and reversible, we get the following power law relationship:

\[ \frac{p_2}{p_1} = \left( \frac{T_2}{T_1} \right)^{\gamma/\gamma-1} = \left( \frac{\rho_2}{\rho_1} \right)^{1/(\gamma-1)} \]

### Control Volume Analysis of a Finite Strength Pressure Wave

<table>
<thead>
<tr>
<th>Moving Wave of Frontal Area A</th>
<th>Stationary Wave</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho )</td>
<td>( \rho )</td>
</tr>
<tr>
<td>( \rho )</td>
<td>( \rho )</td>
</tr>
<tr>
<td>( T )</td>
<td>( T + \Delta T )</td>
</tr>
<tr>
<td>( \Delta V )</td>
<td>( cV )</td>
</tr>
</tbody>
</table>

#### Steady State Continuity Equation (Solve for the induced velocity \( \Delta V \)):

\[ 0 = \int_{c} \rho \left( \nabla \cdot \mathbf{v} \right) \, dA = \int_{A} \rho \, dA + \int_{A} \left( \rho + \Delta \rho \right) (c - \Delta V) \, dA \]

\[ \rho c A = \left( \rho + \Delta \rho \right) (c - \Delta V) A \]

\[ \rho c = c (\rho + \Delta \rho) - \Delta V (\rho + \Delta \rho) \]

\[ \Delta V = \frac{\rho \Delta \rho}{\rho + \Delta \rho} \quad (A) \]

The speed of sound \( c \) is the rate of propagation of a pressure wave of infinitesimal strength through a still fluid.

### Control Volume Analysis of a Finite Strength Pressure Wave

#### Steady State Momentum Equation:

(Find \( \Delta p \) and \( c \))

\[ \sum F_i = \int_{A} \rho V_i \left( \mathbf{v} \cdot \mathbf{n} \right) \, dA = \rho \left( \mathbf{V}_2 - \mathbf{V}_1 \right) \]

\[ pA - (p + \Delta p)A = \rho c (c - \Delta V - c) \]

\[ \Delta p = \rho c \Delta V \quad \text{(B)} \]

Now combine A & B and solve for the speed of sound:

\[ c^2 = \frac{\Delta p}{\Delta \rho} \frac{\rho + \Delta \rho}{\rho} = \frac{\rho c^2}{\rho + \Delta \rho} \left( 1 + \frac{\rho c^2}{\rho} \right) \]

\[ c^2 = \frac{\Delta p}{\Delta \rho} \quad \text{in the limit of } \Delta \rho \to 0 \]

Small amplitude moderate frequency waves are isentropic and

\[ \frac{\rho c^2}{\rho} = c^2 \text{exct} \]
Control Volume Analysis of a Finite Strength Pressure Wave

Calculating the Speed of Sound for an ideal gas:
\[
\frac{\partial P}{\partial P} = const \\
\frac{\partial P}{\partial P} = k \frac{P}{\rho} \quad \Rightarrow \quad c = \sqrt{\frac{k}{\rho} \frac{P}{\rho} = \sqrt{RT}}
\]

For Air:
\[
k = \frac{c}{c} = 1.4 \\
R = 573 J/(kg \cdot K)
\]

Typical Speeds of Sound

<table>
<thead>
<tr>
<th>Fluid</th>
<th>c (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gases:</td>
<td></td>
</tr>
<tr>
<td>H\textsubscript{2}</td>
<td>1,294</td>
</tr>
<tr>
<td>Air</td>
<td>340</td>
</tr>
<tr>
<td>Liquids:</td>
<td></td>
</tr>
<tr>
<td>Water</td>
<td>1,490</td>
</tr>
<tr>
<td>Ethyl Alcohol</td>
<td>1,200</td>
</tr>
</tbody>
</table>

Data From White 2003

Example 1: Speed of sound calculation

Determine the speed of sound in Argon (Ar) at 120 °C. MW = 40 kg/kmol:

\[
c = \sqrt{\frac{kR}{\rho}} \\
R = \frac{8314 J/(kmol \cdot K)}{40 kg/kmol} = 207.91 J/(kg \cdot K) \\
k = \frac{c}{c} = 1.668 \\
c = \sqrt{1.668 \cdot 207.91 J/kgK \cdot 393 K} = 318.8 \text{ms}^{-1}
\]

Movement of a sound source and wave propagation

Source moves to the right at a speed V
Example 2: a needle nose projectile traveling at a speed of $M=3$ passes 200m above an observer. Find the projectiles velocity and determine how far beyond the observer the projectile will first be heard.

\[ c = \sqrt{RT} = \sqrt{1.4(287)(300)} = 347.2\text{m/s} \]
\[ V = Mc = 3(347.2) = 1041.6\text{m/s} \]
\[ \alpha = \sin^{-1}\left(\frac{1}{M}\right) = \sin^{-1}\left(\frac{1}{3}\right) = 19.5^\circ \]
\[ \tan \alpha = \frac{200\text{m}}{x} \]
\[ x = \frac{200\text{m}}{\tan 19.5} = 565\text{m} \]

Steady Isentropic Flow – Control Volume Analysis

Applications where the assumptions of steady, uniform, isentropic flow are reasonable:

1. Exhaust gasses passing through the blades of a turbine.
2. Diffuser near the front of a jet engine
3. Nozzles on a rocket engine
4. A broken natural gas line
Steady Isentropic Flow

Steady State Continuity Equation:

\[
0 = \oint \rho \vec{V} \cdot \vec{n} \, dA = -\rho V_i A_i + \rho V_f A_f
\]

\[
\rho VA = (\rho + d\rho) \gamma V + dV \gamma (A + dA)
\]

\[
\rho VA = \rho A V + \rho V \gamma dA + V \gamma dA \rho + V \gamma dA \rho + \rho \gamma dA V + dA \rho V dA V + dA \rho V dA V + dA \rho V dA V
\]

Only retain 1st order differential terms & divide by \( \rho VA \)

Steady State Energy Equation with 1 inlet & 1 exit:

Neglecting potential energy and recalling:

\[
\frac{Q - W}{m} = \frac{V_f^2 - V_i^2}{2} + \gamma (z_f - z_i) + (\gamma - 1) \left( g f - g i \right)
\]

Assuming ideal gas:

\[
\frac{Q - W}{m} = \frac{V_f^2 - V_i^2}{2} + c_r (T_f - T_i)
\]
### Steady Isentropic Flow

Steady State Energy Equation with 1 inlet & 1 exit, neglecting potential energy & assuming Isentropic duct flow:

\[ \frac{V_{1}^2}{2} + h_z = \frac{V_{2}^2}{2} + h_i \]

Assuming and ideal gas:

\[ \frac{V_{1}^2}{2} + c_p T_i = \frac{V_{2}^2}{2} + c_p T_i \]

\[ \frac{V_{2}^2}{2} + \frac{k}{k-1} R T_i = \frac{V_{1}^2}{2} + \frac{k}{k-1} R T_i \]

### Stagnation Conditions

Assume the area \( A_2 \) is so big \( V_2 \approx 0 \), then

\[ h_2 = \frac{V_{2}^2}{2} + h_i = h \]

Stagnation enthalpy

Similarly, as we adiabatically bring a fluid parcel to zero velocity there is a corresponding increase in temperature

\[ \frac{V_{2}^2}{2} + c_p T_2 = \frac{V_{1}^2}{2} + c_p T_i \]

\[ T_2 = \frac{V_{1}^2}{2c_p} + T \]

Stagnation Temperature

### Stagnation Conditions – maximum velocity

\[ T_0 = \frac{V_{2}^2}{2c_p} + T \quad (+) \]

If the temperature, \( T \) is taken taken down to absolute zero, then (+) can be solved for the maximum velocity:

\[ V_{\text{max}} = \sqrt{2c_p T_0} \]

No higher velocity is possible unless energy is added to the flow through heat transfer or shaft work.
Stagnation Conditions – Mach number relations

Recall, that the Mach number is defined as: \( M = \frac{V}{c} \)

\[
T_o = \frac{V^2}{2c^2} + T
\]

For Ideal gases:

\[
T_o = \frac{V^2}{2c^2} + 1 \quad \to \quad c_pT_o = \left( \frac{kR}{k-1} \right)T_o = \frac{1}{\kappa}RT_o \quad \frac{1}{c_p^2}
\]

\[
T_o = \frac{k-1}{2} \frac{V^2}{c^2} + 1 \quad \Rightarrow \quad \frac{T_o}{T} = \frac{k-1}{2} M^2 + 1
\]

Stagnation Conditions – Isentropic pressure & density relationships

\[
\frac{T_o}{T} = \frac{k-1}{2} M^2 + 1
\]

\[
\frac{p_o}{p} = \left( \frac{T_o}{T} \right)^{\frac{1}{k-1}} = \left( \frac{k-1}{2} M^2 + 1 \right)^{\frac{1}{k-1}}
\]

\[
\frac{p_o}{\rho_o} = \left( \frac{T_o}{T} \right)^{\frac{1}{k}} = \left( \frac{k-1}{2} M^2 + 1 \right)^{\frac{1}{k}}
\]

Critical Values: conditions when \( M = 1 \)

\[
\frac{T^*}{T_o} = \left( \frac{2}{k+1} \right)
\]

\[
\frac{p^*}{p_o} = \left( \frac{2}{k+1} \right)^{\frac{1}{k}}
\]

\[
\frac{p^*}{\rho_o} = \left( \frac{2}{k+1} \right)^{\frac{1}{k}}
\]

\[
\frac{c^*}{c_o} = \left( \frac{2}{k+1} \right)^{\frac{1}{2}}
\]
Critical Values: conditions when $M = 1$

For Air $k = 1.4$

\[
\frac{T'}{T_o} = \left( \frac{2}{k+1} \right)^{\frac{k-1}{k}} = 0.8333
\]

\[
\frac{p'}{p_o} = \left( \frac{2}{k+1} \right)^{\frac{k}{k-1}} = 0.5283
\]

\[
\frac{T'}{T_o} = \left( \frac{2}{k+1} \right)^{\frac{k-1}{k}} = 0.9129
\]

\[
\frac{c'}{c_o} = \left( \frac{2}{k+1} \right)^{\frac{1}{2}} = 0.9129
\]

In all isentropic flow, all critical values are constant.

Critical Values: conditions when $M = 1$

Critical Velocity: is the speed of sound $c^*$

\[
c' = c_o \left( \frac{2}{k+1} \right)^{\frac{1}{2}}
\]

\[
V' = c^* = \sqrt{kRT} = c_o \left( \frac{2}{k+1} \right)^{\frac{1}{2}} \left( \frac{kRT}{k+1} \right)^{\frac{1}{2}}
\]

Example 3: Stagnation Conditions

Air flows adiabatically through a duct. At point 1 the velocity is 240 m/s, with $T_1 = 320K$ and $p_1 = 170kPa$. Compute

(a) $T_o$
(b) $P_o$
(c) $r_o$
(d) $M$
(e) $V_{max}$
(f) $V^*$
Steady Isentropic Duct Flow

Recall, for Steady isentropic flow Continuity:

$$\rho \frac{dV}{dA} = 0$$

For compressible, isentropic flow the momentum equation is:

$$\rho \frac{dV}{dA} = \frac{dp}{dV}$$

Bernoulli’s Equation!

Substitute (†) into (*)

$$\rho \frac{dV}{dA} = \frac{dp}{dV}$$

Recall that the speed of sound is:

$$c^2 = \frac{dp}{dV}$$

Describes how the pressure behaves in nozzles and diffusers under various flow conditions

Nozzle Flow Characteristics

$$\frac{dA}{A} = \frac{dp}{\rho V^2 (1 - M^2)}$$
Steady Isentropic Duct Flow – Nozzles
Diffusers and Converging Diverging Nozzles

\[
\frac{\Delta A}{A} = -\frac{\rho \nabla V}{V^2} (1 - M^2)
\]

Describes how the pressure behaves in nozzles and diffusers under various flow conditions.

Recall, the momentum equation here is:

\[
0 = \frac{\rho}{V} + \nabla V \cdot dA \quad \quad \quad \frac{\rho}{V} = -\nabla V \quad (**)
\]

Now substitute (***) into (††):

\[
\frac{\Delta A}{A} = \frac{\Delta V}{V} (M^2 - 1)
\]

Or,

\[
\frac{\Delta A}{\Delta V} = \frac{A}{V} (M^2 - 1)
\]

Nozzle Flow Characteristics

\[
\frac{\Delta A}{A} = \frac{\Delta V}{V} (M^2 - 1)
\]

1. Subsonic Flow: \( M < 1 \) and \( \Delta A < 0 \), then \( \Delta V > 0 \): indicating an accelerating flow in a converging channel.

2. Supersonic Flow: \( M > 1 \) and \( \Delta A < 0 \), then \( \Delta V < 0 \): indicating a decelerating flow in a converging channel.

3. Subsonic Flow: \( M < 1 \) and \( \Delta A > 0 \), then \( \Delta V < 0 \): indicating a decelerating flow in a diverging channel.

4. Supersonic Flow: \( M > 1 \) \( \Delta A > 0 \), then \( \Delta V > 0 \): indicating an accelerating flow in a diverging channel.

Converging-Diverging Nozzles

Flow can not be sonic
Choked Flow – The maximum possible mass flow through a duct occurs when its throat is at the sonic condition. Consider a converging Nozzle:

\[ p_c, T_c, \rho_c \quad \text{plenum} \quad p_r, V_r \quad \text{receiver} \]

Mass Flow Rate (ideal gas):
\[
\dot{m} = \rho V A = \frac{p V}{RT} \quad \quad M = \frac{V}{c} = \frac{\sqrt{\gamma RT}}{\sqrt{\gamma RT}}
\]
\[
\dot{m} = \frac{p M}{RT M} \quad A = p \left( \frac{k}{RT MA} \right)
\]
\[
\dot{m} = p \left( \sqrt{\frac{k}{RT MA}} \right)
\]

Choked Flow

Mass Flow Rate (ideal gas):
\[
\dot{m} = p \left( \sqrt{\frac{k}{RT MA}} \right)
\]
Recall, the stagnation pressure and temperature ratio and substitute:
\[
\frac{p_c}{p} = \left( \frac{k-1}{2} M^2 + 1 \right)^{\frac{k}{k-1}} \quad \frac{T_c}{T} = \frac{k-1}{2} M^2 + 1
\]
\[
\dot{m} = p_c A \left( \frac{k}{k+1} \right)^{\frac{k+1}{k-1}}
\]
If the critical area \((A^*)\) is where \(M=1\):
\[
\dot{m} = p_c A \left( \frac{k}{k+1} \right)^{\frac{k+1}{k-1}}
\]
The critical area ratio is:
\[
\frac{A}{A^*} = \left( \frac{2 + (k-1)M^2}{k+1} \right)^{\frac{k+1}{k-1}}
\]