Given: Velocity field $\vec{V} = ax(1+bt)\hat{i} + cy\hat{j}$, where $a = c = 1\,s^{-1}$, $b = 0.2\,s^{-1}$, and coordinates are measured in meters

Plot: the streakline that passes through the point $(x_0, y_0) = (1,1)$ during the interval $0 \le t \le 3s$. Compare with the streamlines plotted through the same point at $t = 0$, 1, and 2s

Solution:

Streakline at $t = 3s$ connects particles that passed through point (x_0, y_0) at earlier times $\tau = 0, 1, 2,$ and 3s.

For a particle, $u = dx/dt$ and $v = dy/dt$

Then $u = ax(1+bt) = \frac{dx}{dt}$ and $\int_{x_0}^{x} \frac{dx}{x} = \int_{\tau}^{t} a(1+bt)\,dt$

$\ln \frac{x}{x_0} = a\left(t + \frac{b}{2}t^2\right)_{\tau}^{t} = a\left[(t-\tau) + \frac{b}{2}(t^2 - \tau^2)\right]$

$x = x_0\, e^{a\left[(t-\tau) + \frac{b}{2}(t^2-\tau^2)\right]}$

Also $v = \frac{dy}{dt} = cy$, $\int_{y_0}^{y} \frac{dy}{y} = \int_{\tau}^{t} c\,dt$, $\ln \frac{y}{y_0} = c(t-\tau)$, $y = y_0\, e^{c(t-\tau)}$

Substituting for a, b, c, x_0, and y_0, gives:

$x = e^{\left[(t-\tau) + 0.1(t^2-\tau^2)\right]}$, $y = e^{(t-\tau)}$ ← (x,y) streakline

The streakline may be plotted by substituting values for τ in the range $0 \le \tau \le 3s$ as shown below.

The streamline is found (at given t) from $(dy/dx)_s = v/u$

Thus $dy/dx = \frac{cy}{ax(1+bt)}$ and $\int_{y_0}^{y} \frac{dy}{y} = \int_{x_0}^{x} \frac{c}{a(1+bt)} \frac{dx}{x}$

$\ln \frac{y}{y_0} = \frac{c}{a(1+bt)} \ln \frac{x}{x_0}$ or $y = y_0 \left[\frac{x}{x_0}\right]^{c/a(1+bt)}$

Substituting values for x_0, y_0, a, b, c, then

$y = x^{1/(1+0.2t)}$ or $x = y^{(1+0.2t)}$ ← streamline

At $t = 0$, $x = y_{1.0}$

$t = 1s$, $x = y_{1.2}$

$t = 2s$, $x = y_{1.4}$