Problem 5.42

Given: Incompressible, two-dimensional flow field with $w = 0$, has a y component of velocity given by

$$v = -Axy$$

where units of v are m/s; x and y are in meters and A is a dimensional constant

Find: (a) the dimensions of the constant A
(b) the simplest x component of velocity for this flow field
(c) the acceleration of a fluid particle at the point $(x,y) = (1,2)$

Solution:

(a) Since $v = -Axy$, then the dimensions of A, $[A]$, are given by

$$[A] = \left[\frac{v}{xy}\right] = \frac{L}{t}\cdot\frac{1}{L}\cdot\frac{1}{L} = \frac{1}{Lt} \qquad \boxed{A}$$

(b) Apply the continuity equation for the conditions given

Basic equation: $\nabla \cdot \rho\vec{V} + \frac{\partial \rho}{\partial t} = 0$

For incompressible flow, $\frac{\partial \rho}{\partial t} = 0$. Thus with $w = 0$, the basic equation reduces to $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

Then, $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} = -\frac{\partial}{\partial y}(-Axy) = Ax$

and

$$u = \int \frac{\partial u}{\partial x}dx + f(y) = \int Ax\,dx + f(y) = \frac{1}{2}Ax^2 + f(y)$$

The simplest x component of velocity is obtained with $f(y) = 0$

$$\therefore u = \frac{1}{2}Ax^2 \qquad u$$

(c) The acceleration of a fluid particle is given by

$$\vec{a}_p = \frac{D\vec{V}}{Dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + \cancel{w\frac{\partial \vec{V}}{\partial z}}^{\,0} + \cancel{\frac{\partial \vec{V}}{\partial t}}^{\,0}$$

$$\vec{a}_p = \frac{1}{2}Ax^2\frac{\partial}{\partial x}\left[\frac{1}{2}Ax^2\hat{i} - Axy\hat{j}\right] - Axy\frac{\partial}{\partial y}\left[\frac{1}{2}Ax^2\hat{i} - Axy\hat{j}\right]$$

$$\vec{a}_p = \frac{1}{2}Ax^2\left[Ax\hat{i} - Ay\hat{j}\right] - Axy\left[-Ax\hat{j}\right] = \frac{1}{2}A^2x^3\hat{i} + \frac{1}{2}A^2x^2y\hat{j}$$

At the point $(x,y) = (1,2)$

$$\vec{a}_p = \frac{1}{2}A^2(1)^3\hat{i} + \frac{1}{2}A^2(1)^2(2)\hat{j} = A^2\left[\frac{1}{2}\hat{i} + \hat{j}\right] \qquad \vec{a}_p$$