Control Volume Forms of the Fundamental Laws

1. Conservation of Mass
2. Conservation of Linear Momentum
3. Conservation of Angular Momentum (moment of momentum)
4. Conservation of Energy

Derivation of the Reynolds Transport Theorem

What: A formal mathematical expression which allows the time rate of change of an extensive property for a given quantity of mass, a system, to be expressed in terms of quantities related to a specific region of space, a control volume.

Why: All conservation laws are written for a system. We need to express the time rate of change for a system in terms of a control volume. That is, since it is difficult to identify and follow the same mass of fluid, we need an Eulerian description.

Definitions

- **System** - A fixed collection of mass particles
- **Control Volume** – defined region in space
- **Extensive Property** \((N)\) – property of the system that depends on mass (“Stuff”).
  - Momentum, internal energy, entropy
- **Intensive Property** \((\eta)\) – property of the system that is independent of mass
  - Temperature, velocity, specific energy
Objective: Describe the rate at which an integral quantity associated with the system is changing as the flow passes into and out of the Control Volume.

Note: Fluid DOES NOT pass in and out of a system.

Integral Properties:
- Mass flow rate: 
  \[ \dot{m} = \oint_{C} \rho \mathbf{V} \cdot \mathbf{n} \, dA \]
- Mass: 
  \[ m = \int_{V} \rho \mathbf{V} \, dV \]
- Drag Force: 
  \[ F_{d} = \oint_{C} \tau \, dA \]
- Kinetic Energy: 
  \[ KE = \oint_{C} \rho \frac{1}{2} \mathbf{V}^2 \, dA \]

Extensive/Intensive Properties:
- Momentum: 
  \[ N = m \mathbf{V} \]
- Mass: 
  \[ N = \int_{V} \rho \mathbf{V} \, dV \]
  \[ \eta = \frac{m}{\dot{m}} = \frac{\mathbf{V}}{\mathbf{\dot{V}}} \]

System
- \[ N_{sys} = \oint_{C_{sys}} \rho \mathbf{V} \, dA \]

Control Volume
- \[ N_{CV} = \oint_{C_{CV}} \rho \mathbf{V} \, dA \]
Reynolds Transport Theorem for a Deformable Control Volume

Region 6 = 2 + A + B

Reynolds Transport Theorem for a Deformable Control Volume

Region 6 = 2 + A + B
Finding the Size of the “Sweeping Volume”

Elemental volume from 2 - Incoming

\[ d\mathbf{V}_i = -\mathbf{v}_i \cdot \mathbf{n} \, dA \]

Elemental volume from 5 - Outgoing

\[ d\mathbf{V}_o = \mathbf{v}_o \cdot \mathbf{n} \, dA \]

\( \mathbf{n} \) = a length normal to the CS

\[ N_i = -\int_{\Delta\mathbf{v}} \rho \eta (\mathbf{v} \cdot \mathbf{n}) dA, \Delta \mathbf{v} \]

\[ N_o = \int_{\Delta\mathbf{v}} \rho \eta (\mathbf{v} \cdot \mathbf{n}) dA, \Delta \mathbf{v} \]

Reynolds Transport Theorem

\[ \frac{dN}{dt} = \int_{\Delta\mathbf{v}} \rho \eta (\mathbf{v} \cdot \mathbf{n}) dA \]

\[ = \int_{\mathbf{v}} \rho \eta (\mathbf{v} \cdot \mathbf{n}) + \int_{\mathbf{\dot{v}}} \rho \eta (\mathbf{v} \cdot \mathbf{n}) dA \]

\( \frac{dN}{dt} \) = time rate of change of system of particles being followed

\( \int_{\mathbf{v}} \rho \eta (\mathbf{v} \cdot \mathbf{n}) \) = flux term

\( \int_{\mathbf{\dot{v}}} \rho \eta (\mathbf{v} \cdot \mathbf{n}) dA \) = time rate of change term

\[ \int_{\Delta\mathbf{v}} \rho \eta (\mathbf{v} \cdot \mathbf{n}) dA, \Delta \mathbf{v} \]