Runge-Kutta Methods – CH 25

Can achieve Taylor Series accuracy without evaluating higher order derivatives.

General form: \( y_{i+1} = y_i + \phi(x_i, y_i, h)h \) \hspace{1cm} (1)

\( \phi(x_i, y_i, h) \) - Increment function & is like a slope over the interval

\[ \phi = a_1 k_1 + a_2 k_2 + \ldots + a_n k_n \]

• a’s are constants & k’s are recurrence relationships
• n=1 \( \rightarrow \) Euler’s method
Runge-Kutta Methods – CH 25

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**General form:** \( y_{i+1} = y_i + \phi(x_i, y_i, h)h \) \hspace{1cm} (1)

\( \phi(x_i, y_i, h) \) - *Increment function* & is like a slope over the interval

\[ \phi = a_1 k_1 + a_2 k_2 + \ldots + a_n k_n \]
\[ k_1 = f(x_i, y_i) \]
\[ k_2 = f(x_i + p_1 h, y_i + q_{11} k_1 h) \]
\[ k_3 = f(x_i + p_2 h, y_i + q_{21} k_1 h + q_{22} k_2 h) \]
\[ k_n = f(x_i + p_{n-1} h, y_i + q_{n-1,1} k_1 h + q_{n-1,2} k_2 h + \ldots + q_{n-1,n-1} k_{n-1} h) \]

* a’s are constants & k’s are recurrence relationships

• n=1 \( \rightarrow \) Euler’s method
Runge-Kutta Methods

To Determine the final form of (1)

1. Select $n$

2. Evaluate $a’s,p’s,q’s$ by setting the general form equal to terms in the T-S expansion.

3. For low-order forms
   - Number of terms $n=$order of the method
   - *Local truncation error* is $O(h^{n+1})$
   - *Global truncation error* is $O(h^n)$
2nd- Order Runge-Kutta Methods

General Form: 

\[ y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)h \]  \hspace{1cm} (2)

\[ k_1 = f(x_i, y_i) \]

\[ k_2 = f(x_i + p_1 h, y_i + q_{11} k_1 h) \]

By setting (2) equal to a T-S expansion through the 2nd order term, we can solve for \( a_1, a_2, p_1, q_{11} \)

\[
\begin{align*}
    a_1 + a_2 &= 1 \\
    a_2 p_1 &= 1/2 \\
    a_2 q_{11} &= 1/2
\end{align*}
\]

\[ \begin{align*}
    a_1 &= 1 - a_2 \\
    p_1 &= 1/(2a_2) \\
    q_{11} &= 1/(2a_2)
\end{align*} \]

Specify \( a_2 \) value

*Since there are an infinite number of choices for \( a_2 \) there will be an infinite number of 2nd order R-K Methods*
2nd-Order Runge-Kutta Methods

A) Huen Method without iteration

\(a_2 = \frac{1}{2}\): \(a_1 = \frac{1}{2}, p_1 = 1, q_{11} = 1\)

\[y_{i+1} = y_i + \left(\frac{1}{2} k_1 + \frac{1}{2} k_2\right) h\]

\[k_1 = f(x_i, y_i)\]

\[k_2 = f(x_i + h, y_i + k_1 h)\]

\(k_1\) slope at start of interval

\(k_2\) slope at end of interval

\[a_1 = 1 - a_2\]

\[p_1 = 1/(2a_2)\]

\[q_{11} = 1/(2a_2)\]

Global Truncation Error \(\sim O(h^2)\)
2nd- Order Runge-Kutta Methods

B) Midpoint Method \((a_2 = 1)\): \(a_1 = 0\), \(p_1 = 1/2\), \(q_{11} = 1/2\)

\[ y_{i+1} = y_i + k_2 h \]

\[ k_1 = f(x_i, y_i) \]
\[ k_2 = f(x_i + 0.5h, y_i + 0.5k_1 h) \]

\[ a_1 = 1 - a_2 \]
\[ p_1 = 1/(2a_2) \]
\[ q_{11} = 1/(2a_2) \]

*Global Truncation Error* \(\sim O(h^2)\)
2nd- Order Runge-Kutta Methods

C) Ralston’s Method \( (a_2 = 2/3 )\): \( a_1 = 1/3 \), \( p_1 = 3/4 \), \( q_{11} = 3/4 \)

\[
y_{i+1} = y_i + \left( \frac{1}{3} k_1 + \frac{2}{3} k_2 \right) h
\]

\[
k_1 = f(x_i, y_i)
\]

\[
k_2 = f(x_i + 0.75h, y_i + .75k_1h)
\]

\[
a_1 = 1 - a_2
\]

\[
p_1 = 1/(2a_2)
\]

\[
q_{11} = 1/(2a_2)
\]

Global Truncation Error \( \sim O(h^2) \)
4th - Order Runge-Kutta Methods

Classical 4th order RK Method – most commonly used RK method

\[ y_{i+1} = y_i + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)h \]

\[ y_{i+1} = y_i + \phi h \]

Slope Estimates:

\[ k_1 = f(x_i, y_i) \]
\[ k_2 = f(x_i + 0.5h, y_i + 0.5k_1h) \]
\[ k_3 = f(x_i + 0.5h, y_i + 0.5k_2h) \]
\[ k_4 = f(x_i + h, y_i + k_3h) \]

Global Truncation Error \( \sim O(h^4) \)
4th - Order Runge-Kutta Methods –

Example: Use classical RK4 to determine $y @ x=0.4$ for $y'=x-y$ and $h=0.4$

Recall the exact solution is:

$$y = x + e^{-x} - 1$$

$$y(0.4) = 0.070320$$

RK4 Solution:

$$x_0 = 0$$
$$y_0 = 0$$

Initial Conditions

$$k_1 = f(x_i, y_i) = x_0 - y_0 = 0$$

$$k_2 = f(x_i + 0.5h, y_i + .5k_1h) = (0 + 0.4 / 2) - (0 + 0) = 0.2$$

$$k_3 = f(x_i + 0.5h, y_i + .5k_2h) = (0 + 0.4 / 2) - (0 + (0.5)(0.2)(0.4)) = 0.16$$

$$k_4 = f(x_i + 0.5h, y_i + k_3h) = (0 + 0.4) - (0 + 0.16(0.4)) = 0.336$$

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h = 0 + \frac{1}{6}(0 + 2(.2) + 2(.16) + .336)0.4$$

$$y_1 = 0.07040$$
Example: Use classical RK4 to determine $y @ x=0.4$ for $y' = x - y$ and $h = 0.4$

**Error:**

$$E_t = \left| \frac{.07032 - .07040}{.07032} \right| \times 100\% = .11\%$$

See Matlab Sample Matlab RK4 method
Method Comparison

• Higher order methods produce better accuracy
• Effort for the higher order methods is similar to low-order methods (much of the effort goes into evaluating the function)
• Classical 4\textsuperscript{th} order RK is most widely used as it produces accurate results with reasonable effort.
Systems of Equations

• Recall, Any $n^{th}$ order ODE can be represented as a system of $n$ 1$^{st}$ order ODEs

$$\frac{dy_1}{dx} = f_1(x, y_1, y_2, \ldots, y_n)$$

$$\frac{dy_2}{dx} = f_2(x, y_1, y_2, \ldots, y_n)$$

$$\vdots$$

$$\frac{dy_n}{dx} = f_n(x, y_1, y_2, \ldots, y_n)$$

• To solve the system requires $n$ initial conditions at $x = x_0$
Systems of Equations – RK4 Example

\[
\frac{dy_1}{dx} = f_1(x, y_1, y_2)
\]
\[
\frac{dy_2}{dx} = f_2(x, y_1, y_2)
\]

For example:

\[
\frac{dy}{dx} = f_1(x, y, z) = -y
\]
\[
\frac{dz}{dx} = f_2(x, y, z) = 3 - 4\cos z + y
\]

Subject to initial conditions
\[
y_{1,0} = y_1(x = 0) = Y_1
\]
\[
y_{2,0} = y_2(x = 0) = Y_2
\]
Systems of Equations – RK4 Example

Solve for slopes

\[ k_{i,j} \]

\( i \)th value of \( k \) for the \( j \)th dependant variable

For RK-4 \( i=1, 2, 3 \) and 4 while \( j=1, 2, \ldots \) number of dependant variables
Systems of Equations – RK4 Example

Solve for slopes
Start with $i=0$
The initial condition

\[ k_{1,1} = f_1(x_i, y_{1i}, y_{2i}) \]
\[ k_{1,2} = f_2(x_i, y_{1i}, y_{2i}) \]
\[ k_{2,1} = f_1\left(x_i + \frac{1}{2} h, y_{1i} + \frac{1}{2} k_{11} h, y_{2i} + \frac{1}{2} k_{12} \right) \]
\[ k_{2,2} = f_2\left(x_i + \frac{1}{2} h, y_{1i} + \frac{1}{2} k_{11} h, y_{2i} + \frac{1}{2} k_{12} h \right) \]
\[ k_{3,1} = f_1\left(x_i + \frac{1}{2} h, y_{1i} + \frac{1}{2} k_{21} h, y_{2i} + \frac{1}{2} k_{22} h \right) \]
\[ k_{3,2} = f_2\left(x_i + \frac{1}{2} h, y_{1i} + \frac{1}{2} k_{21} h, y_{2i} + \frac{1}{2} k_{22} h \right) \]
\[ k_{4,1} = f_1\left(x_i + h, y_{1i} + k_{31} h, y_{2i} + k_{32} h \right) \]
\[ k_{4,2} = f_2\left(x_i + h, y_{1i} + k_{31} h, y_{2i} + k_{32} h \right) \]
Systems of Equations – RK4 Example

\[
y_{1,i+1} = y_{1i} + \frac{1}{6} (k_{11} + 2k_{21} + 2k_{31} + k_{41})h
\]

\[
y_{2,i+1} = y_{2i} + \frac{1}{6} (k_{12} + 2k_{22} + 2k_{32} + k_{42})h
\]

Show Matlab Systems of Equations RK4 Example

\[
\frac{dy_1}{dx} = y_2
\]

\[
\frac{dy_2}{dx} = -\frac{y_2}{2} - 7y_1
\]

\[
y_1(x = 0) = 4
\]

\[
y_2(x = 0) = 0
\]
Matlab ODE solvers

ODE23 and ODE45 are RK solvers that combine 2\textsuperscript{nd} and 3\textsuperscript{rd} order RK and 4\textsuperscript{th} and 5\textsuperscript{th} order RK methods.

See Chapter 8 in Palm Text.