Solution Key: Homework 1

1. Use Gauss' Theorem ($\nabla \cdot \vec{B} = \lim_{V \to 0} \frac{1}{V} \int_S \vec{B} \cdot \hat{n} dS$) to show that for a cubical volume element in Cartesian coordinates that $\nabla \cdot \vec{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}$.

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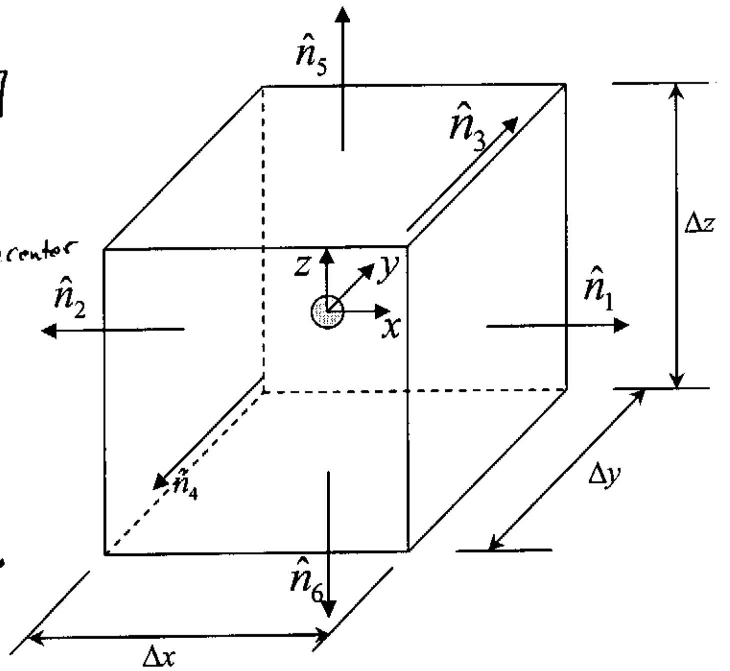
$$\frac{1}{V}\int_S \vec{B}\cdot\hat{n}\,ds = \frac{1}{V}\Big[\int_1 \vec{B}\cdot\hat{n}\,ds + \int_2 \vec{B}\cdot\hat{n}\,ds + \int_3 \vec{B}\cdot\hat{n}\,ds + \int_4 \vec{B}\cdot\hat{n}\,ds + \int_5 \vec{B}\cdot\hat{n}\,ds + \int_6 \vec{B}\cdot\hat{n}\,ds\Big]$$



As $\Delta x, \Delta y, \Delta y \to 0$ the surface integrals can be approvimated by the product of B at the center and the surface area ΔS.

For the coordinate system shown:

$\hat{n}_1 = \hat{i}$ $\quad \hat{n}_3 = \hat{j}$ $\quad \hat{n}_5 = \hat{k}$

$\hat{n}_2 = -\hat{i}$ $\quad \hat{n}_4 = -\hat{j}$ $\quad \hat{n}_6 = -\hat{k}$

$$\int_1 \vec{B}\cdot\hat{n}\,ds = \left(B_x(x+\tfrac{\Delta x}{2},y,z)\hat{i} + B_y(x+\tfrac{\Delta x}{2},y,z)\hat{j} + B_z(x+\tfrac{\Delta x}{2},y,z)\right)\cdot(\hat{i}+0\hat{j}+0\hat{k})\,\Delta y\Delta z$$

$$= B_x(x+\tfrac{\Delta x}{2},y,z)\,\Delta y\Delta z$$

$$\int_2 \vec{B}\cdot\hat{n}\,ds = \left(B_x(x-\tfrac{\Delta x}{2},y,z)\hat{i} + B_y(x-\tfrac{\Delta x}{2},y,z)\hat{j} + B_z(x-\tfrac{\Delta x}{2},y,z)\right)\cdot(-\hat{i}+0\hat{j}+0\hat{k})\,\Delta y\Delta z$$

$$= -B_x(x-\tfrac{\Delta x}{2})\,\Delta y\Delta z$$

$$\int_3 \vec{B}\cdot\hat{n}\,ds = B_y(x,y+\tfrac{\Delta y}{2},z)\,\Delta x\Delta z$$

$$\int_4 \vec{B}\cdot\hat{n}\,ds = -B_y(x,y-\tfrac{\Delta y}{2},z)\,\Delta x\Delta z$$

$$\int_5 \vec{B}\cdot\hat{n}\,ds = B_z(x,y,z+\tfrac{\Delta z}{2})\,\Delta x\Delta y$$

$$\int_6 \vec{B}\cdot\hat{n}\,ds = -B_z(x,y,z-\tfrac{\Delta z}{2})\,\Delta x\Delta y$$

$$\frac{1}{V}\int \vec{B}\cdot\hat{n}\,ds = \frac{1}{\Delta x\Delta y\Delta z}\Big[\left(B_x(x+\tfrac{\Delta x}{2}) - B_x(x-\tfrac{\Delta x}{2},y,z)\right)\Delta y\Delta z + \left(B_y(x,y+\tfrac{\Delta y}{2},z) - B_y(x,y-\tfrac{\Delta y}{2},z)\right)\Delta x\Delta z + \left(B_z(x,y,z+\tfrac{\Delta z}{2}) - B_z(x,y,z-\tfrac{\Delta z}{2})\right)\Delta x\Delta y$$

$$= \frac{B_x(x+\frac{\Delta x}{2},y,z) - B_x(x-\frac{\Delta x}{2},y,z)}{\Delta x} + \frac{B_y(x,y+\frac{\Delta y}{2},z) - B_y(x,y-\frac{\Delta y}{2},z)}{\Delta y} + \frac{B_z(x,y,z+\frac{\Delta z}{2}) - B_z(x,y,z-\frac{\Delta z}{2})}{\Delta z}$$

take limit $\Delta x, \Delta y\, \Delta z \to 0$

$$\lim_{V\to 0}\frac{1}{V}\int \vec{B}\cdot\hat{n}\,dA = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = \vec{\nabla}\cdot\vec{B}$$

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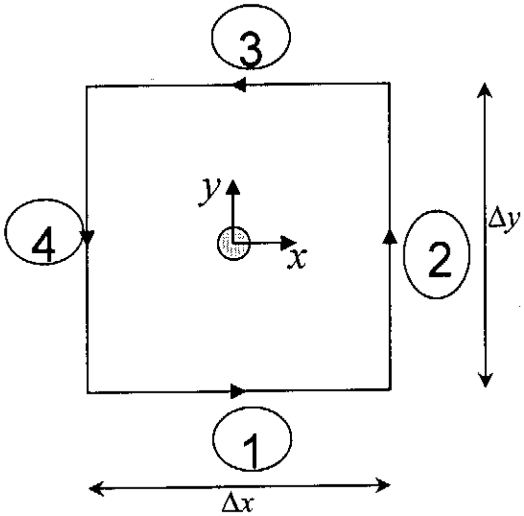
2. Stokes' Theorem

a. Use Stokes' theorem ($\left(\left(\nabla \times \vec{B}\right) \cdot \hat{n} = \lim_{S \to 0} \frac{1}{S} \oint \vec{B} \cdot d\vec{l}\right.$, where S is the surface area) in Cartesian coordinates for an elemental rectangle to show that the z-component of the curl of the vector $\vec{V} = u_x \hat{i} + u_y \hat{j} + u_z \hat{k}$ is given by $\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}$.



$d\vec{l} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

$\oint \vec{V} \cdot d\vec{l} = \int_1 \vec{V} \cdot d\vec{l} + \int_2 \vec{V} \cdot d\vec{l} + \int_3 \vec{V} \cdot d\vec{l} + \int_4 \vec{V} \cdot d\vec{l}$

$\int_1 \vec{V} \cdot d\vec{l}_1 = \left(u_x(x, y - \frac{\Delta y}{2})\hat{i} + u_y(x, y - \frac{\Delta y}{2})\hat{j}\right) \cdot \left(\Delta x \hat{i} + \Delta y \hat{j}\right)$

$= u_x(x, y - \frac{\Delta y}{2})\Delta x$

$\int_2 \vec{V} \cdot d\vec{l}_2 = u_y(x + \frac{\Delta x}{2}, y)\Delta y$

$\int_3 \vec{V} \cdot d\vec{l}_3 = -u_x(x, y + \frac{\Delta y}{2})\Delta x$

$\int_4 \vec{V} \cdot d\vec{l}_4 = -u_y(x - \frac{\Delta x}{2})\Delta y$

$\Delta \vec{l}_1 = \Delta x\,\hat{i}$
$\Delta \vec{l}_2 = \Delta y\,\hat{j}$
$\Delta \vec{l}_3 = -\Delta x\,\hat{i}$
$\Delta \vec{l}_4 = -\Delta y\,\hat{j}$

$\frac{1}{S} \int \vec{V} \cdot d\vec{l} = \frac{1}{\Delta x \Delta y}\left[\left(u_y(x + \frac{\Delta x}{2}, y) - u_y(x - \frac{\Delta x}{2})\right)\Delta y - \left(u_x(x, y + \frac{\Delta y}{2}) - u_x(x, y - \frac{\Delta y}{2})\right)\Delta x\right]$

Divide by $\Delta x \Delta y$
take lim $\Delta x, \Delta y \to 0$

$$\frac{u_y(x + \frac{\Delta x}{2}, y) - u_y(x - \frac{\Delta x}{2}, y)}{\Delta x} - \frac{u_x(x, y + \frac{\Delta y}{2}) - u_x(x, y - \frac{\Delta y}{2})}{\Delta y}$$

$$\boxed{= \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}}$$

1 b. $\vec{\nabla} \times \vec{\nabla} \phi = 0$

4. $$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

$$\eta = \frac{y}{\sqrt{4\nu t}} = y(4\nu t)^{-1/2}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \eta} \cdot \frac{\partial \eta}{\partial t} \qquad \frac{\partial \eta}{\partial t} = -\frac{y}{\cancel{2}}(4\nu t)^{-3/2}\,\cancel{4}^{2}\nu$$

$$= -\frac{2\cancel{\nu}}{4\cancel{\nu}t} \cdot \frac{y}{(4\nu t)^{1/2}} = \frac{-y}{2\cancel{t}(4\nu t)^{1/2}} = -\frac{1}{2}\frac{\eta}{t}$$

$$\frac{\partial u}{\partial t} = \left(-\frac{1}{2}\frac{\eta}{t}\right)\frac{\partial \eta}{\partial t}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) \Rightarrow \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{\partial u}{\partial \eta}(4\nu t)^{-1/2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial \eta}\cdot(4\nu t)^{1/2}\right) = \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \eta}(4\nu t)^{1/2}\right)\frac{\partial \eta}{\partial y}$$

$$= \frac{\partial^2 u}{\partial \eta^2}\frac{1}{4\nu t} =$$

$$\frac{\partial \eta}{\partial t}\left(\frac{-\eta}{\cancel{2}t}\right) = \frac{\cancel{\nu}}{2\cancel{4}\cancel{\nu}t}\frac{\partial^2 u}{\partial \eta^2}$$

$$\boxed{\frac{\partial \eta}{\partial t} = -\frac{1}{2\eta}\frac{\partial^2 u}{\partial \eta^2}}$$

5.

(a) $\vec{\nabla}\cdot\vec{v} = \dfrac{\partial u_i}{\partial x_i}$

(b) $(\vec{v}\cdot\vec{\nabla})\alpha = u_j \dfrac{\partial \alpha}{\partial x_j}$

(c) $\delta_{i2} F_i$

(2)

6. (a) Show $\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$

$$\vec{D} = \vec{B}\times\vec{C}$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\times\vec{D})_m = \varepsilon_{pqm} A_p D_q$$

$$(\vec{B}\times\vec{C})_q = \varepsilon_{ijq}\, b_i c_j$$

$$\therefore\ \vec{A}\times(\vec{B}\times\vec{C}) = \varepsilon_{pqm} A_p [\varepsilon_{ijq}\, b_i c_j]$$

$$= -\varepsilon_{ijq}\varepsilon_{qpm}\, a_p b_i c_j$$

<u>but</u> $\varepsilon_{ijq}\varepsilon_{qpm} = (\delta_{ip}\delta_{jm} - \delta_{im}\delta_{jp})$

$$-\varepsilon_{ijq}\varepsilon_{qpm} = -(\delta_{ip}\delta_{jm} - \delta_{im}\delta_{jp})$$

$$-\varepsilon_{ijq}\varepsilon_{qpm}\, a_p b_i c_j = -(\delta_{ip}\delta_{jm} - \delta_{im}\delta_{jp})\, a_p b_i c_j$$

$$= -b_p a_p c_m + b_m c_p a_p$$

6(b) a, b, c are coplanar if (2)

$$(\vec{a}\times\vec{b})\cdot\vec{c} = 0$$

or

$$d_m = (\vec{a}\times\vec{b})_m = \varepsilon_{ijm}\, a_i b_j$$

$$(\vec{a}\times\vec{b})\cdot\vec{c} = \varepsilon_{ijm}\, a_i b_j c_m$$

Exercise 2.1

To show $(\underset{\sim}{a}\times\underset{\sim}{b}\times\underset{\sim}{c}) = (\underset{\sim}{a}\cdot\underset{\sim}{c})\underset{\sim}{b} - (\underset{\sim}{a}\cdot\underset{\sim}{b})\underset{\sim}{c}$ (1)

Let $\underset{\sim}{d} = \underset{\sim}{b}\times\underset{\sim}{c}$. Then m component of left side of (1) is

$$(a\times d)_m = \varepsilon_{pqm}a_p d_q = \varepsilon_{pqm}a_p[\varepsilon_{ijq}\, b_i c_j] = -\varepsilon_{ijq}\varepsilon_{qpm}\, b_i c_j a_p$$

$$= -(\delta_{ip}\delta_{jm} - \delta_{im}\delta_{pj})b_i c_j a_p = -b_p c_m a_p + b_m c_p a_p$$

This is the m component of right side of (1).

Exercise 2.2

$\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are coplanar if $(\underset{\sim}{a}\times\underset{\sim}{b})\cdot\underset{\sim}{c} = 0$. This requires

$$\varepsilon_{ijk}a_i b_j c_k = 0$$

Exercise 2.3

(i) $\delta_{ij}\delta_{ij} = \delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$

(ii) $\varepsilon_{pqr}\varepsilon_{pqr} = \varepsilon_{pqr}\varepsilon_{rpq} = \delta_{pp}\delta_{qq} - \delta_{pq}\delta_{qp} = 3(3) - \delta_{pp}$

$$= 9 - 3 = 6$$

(iii) $\varepsilon_{pqi}\varepsilon_{pqj} = \varepsilon_{ipq}\varepsilon_{pqj} = -\varepsilon_{ipq}\varepsilon_{qpj}$

$$= -(\delta_{ip}\delta_{pj} - \delta_{ij}\delta_{pp}) = -\delta_{ij} + 3\delta_{ij} = 2\delta_{ij}$$

Exercise 2.4

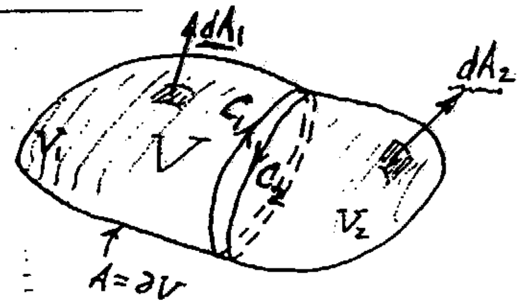
To show $\underset{\sim}{C}\cdot\underset{\sim}{C}^T = \underset{\sim}{C}^T\cdot\underset{\sim}{C} = \underset{\sim}{\delta}$. The original and transformed coordinates (primed) are related by

$$x_j = C_{ji}x'_i \qquad (1)$$

$$x'_j = C_{ij}x_i \qquad (2)$$

From (2) $x'_i = C_{mi}x_m$. Then (1) becomes

Exercise 2.10



Start with the divergence theorem

$$\int_V \nabla \cdot \mathbf{F}\, dV = \int_{A=\partial V} d\mathbf{A} \cdot \mathbf{F}$$

for any vector **F**. Let $\mathbf{F} = \operatorname{curl} \mathbf{u}$.

$$\int_V \operatorname{div}(\operatorname{curl} \mathbf{u})\, dV = \int_{A=\partial V} d\mathbf{A} \cdot \operatorname{curl} \mathbf{u}.$$

Now split the volume into two parts:

$$V = V_1 + V_2, \quad A_1 = \partial V_1, \quad A_2 = \partial V_2$$

$$\int_V \operatorname{div}(\operatorname{curl} \mathbf{u})\, dV = \int_{A_1=\partial V_1} d\mathbf{A} \cdot \operatorname{curl} \mathbf{u} + \int_{A_2=\partial V_2} d\mathbf{A} \cdot \operatorname{curl} \mathbf{u}.$$

A_1 and A_2 have a part of their surfaces that bound the volume V_1 and on those portions $d\mathbf{A}_1$, and $d\mathbf{A}_2$ are oriented normal outwards from V as shown in the sketch. In addition A_1 and A_2 have a *common* boundary surface on the cut. On that common boundary $d\mathbf{A}_1 = -d\mathbf{A}_2$. Thus the sum of the two integrals on the cut surface $= 0$. The remaining portions of A_1 and A_2 are open surfaces such that they sum to $A = \partial V$ and are each in turn bounded by a closed path $C_1 = \partial A_1$ and $C_2 = \partial A_2$, respectively. Using Stokes' theorem,

$$\int_V \operatorname{div}(\operatorname{curl} \mathbf{u})\, dV = \oint_{C_1=\partial A_1} \mathbf{u} \cdot d\mathbf{r} + \oint_{C_2=\partial A_2} \mathbf{u} \cdot d\mathbf{r}.$$

We note that C_1 and C_2 are in fact the same contour traversed in opposite directions (by the right hand rule):

$$C_2 = -C_1 \quad \text{so} \quad \oint_{C_1} + \oint_{C_2} = 0 \text{ and } \int_V \operatorname{div}(\operatorname{curl} \mathbf{u})\, dV = 0$$

for any volume V. For a continuous integrands, we must then have

$$\operatorname{div}(\operatorname{curl} \mathbf{u}) = 0.$$

Exercise 2.11

Stokes' theorem is

$$\int_A \operatorname{curl} \mathbf{F} \cdot d\mathbf{A} = \int_{C=\partial A} \mathbf{F} \cdot d\mathbf{r}$$

for any vector $\mathbf{F}$. Let $\mathbf{F} = \operatorname{grad} \phi$. Now $(\operatorname{grad} \phi) \cdot d\mathbf{r} = d\phi$, so

$$\int_A \operatorname{curl}(\operatorname{grad} \phi) \cdot d\mathbf{A} = \oint_{C=\partial A} d\phi = 0$$

for all single-valued ϕ.