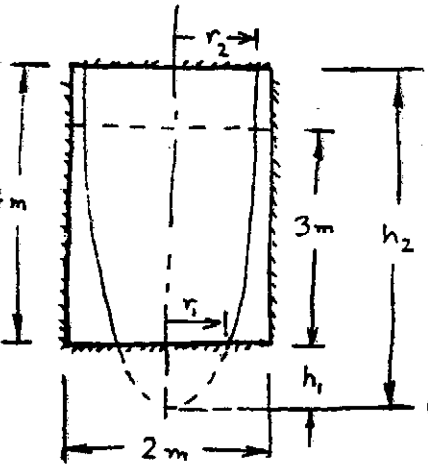
CHAPTER 5



Exercise 5.1

Angular velocity ω = 40 rad/s

$h = \omega^2 r^2/2g$

$h_1 = \omega^2 r_1^2/2g = (40)^2 r_1^2/2(9.81) = 81.5 r_1^2$ (1)

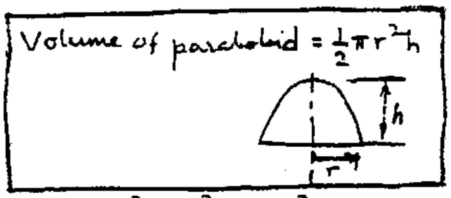
$h_2 = 4 + h_1 = \omega^2 r_2^2/2g = 81.5 r_2^2$ (2)

Subtracting (2) - (1), we get

$$4 = 81.5(r_2^2 - r_1^2) \longrightarrow \qquad r_2^2 - r_1^2 = 0.0491 \text{ m}^2 \qquad (3)$$

Now volume of air within the tank does not change, so

$$\frac{\pi}{4}(2)^2(4-3) = \frac{1}{2}\pi r_2^2 h_2 - \frac{1}{2}\pi r_1^2 h_1$$



or $$2 = h_2 r_2^2 - h_1 r_1^2$$

$$= 81.5(r_2^4 - r_1^4) = 81.5(r_2^2 + r_1^2)(r_2^2 - r_1^2)$$

where we have used (1) and (2) to replace h_1 and h_2. Using (3), the above becomes

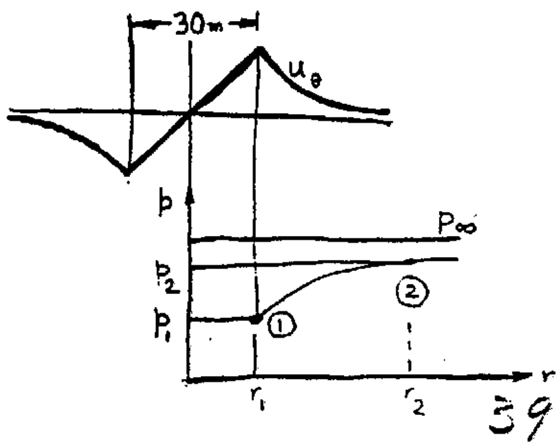
$$r_2^2 + r_1^2 = 2/(81.5)(r_2^2 - r_1^2) = 2/(81.5)(0.0491) = 0.5 \qquad (4)$$

Subtracting (4) - (3), we get $2r_1^2 = 0.5 - 0.0491 = 0.45$. Thus

$$r_1 = 0.475 \text{ m}$$

so that $$\text{Area uncovered} = \pi r_1^2 = 0.71 \text{ m}^2$$

Exercise 5.2



p_1 = - 2000 N/m^2

p_2 = - 500 N/m^2

ρ = 1.18 kg/m^3 at 25°C

$V = 25$ m/s

(a) Applying Bernoulli equation between infinity and edge of core, we get

$$U_1 = \sqrt{2(p_\infty - p_1)/\rho} = \sqrt{2(2000)/1.18} = 58.2 \text{ m/s}$$

$$\therefore \quad \Gamma = 2\pi r_1 U_1 = 2\pi(15)(58.2) = 5485 \text{ m}^2\text{/s}$$

(b) Apply Bernoulli equation between points 2 and 1:

$$p_1 + \tfrac{1}{2}\rho U_1^2 = p_2 + \tfrac{1}{2}\rho U_2^2$$

$$U_2 = \sqrt{2(p_1 - p_2)/\rho + U_1^2} = \sqrt{2(-2000 + 500)/1.18 + 58.2^2}$$

$$= 29.07 \text{ m/s}$$

Since circulation outside core is constant, $r_1 U_1 = r_2 U_2$. So

$$r_2 = r_1 U_1 / U_2 = 30.0 \text{ m}$$

$$\text{Time required} = (r_2 - r_1)/V = (30 - 15)/25 = 0.6 \text{ s}$$

<u>Exercise</u> <u>5.3</u>

Given

$$u_R = 0$$
$$u_\varphi = aRx$$
$$u_x = 0$$

(a) From Appendix B the vorticity components are

$$\omega_R = \frac{1}{R}\cancel{\frac{\partial u_x}{\partial \varphi}} - \frac{\partial u_\varphi}{\partial x} = -aR$$

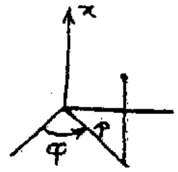
$$\omega_\varphi = \cancel{\frac{\partial u_R}{\partial x}} - \cancel{\frac{\partial u_x}{\partial R}} = 0$$

$$\omega_x = \frac{1}{R}\frac{\partial}{\partial R}(Ru_\varphi) - \frac{1}{R}\cancel{\frac{\partial u_R}{\partial \varphi}} = \frac{1}{R}\frac{\partial}{\partial R}(aR^2x) = 2ax$$

(b)

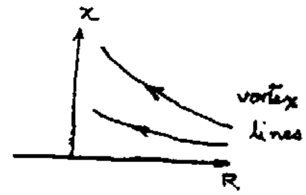
$$\nabla \cdot \omega = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \omega_R}{\partial R}\right) + \frac{1}{R}\frac{\partial \omega_\varphi}{\partial \varphi} + \frac{\partial \omega_x}{\partial x}$$

$$= \frac{1}{R}\frac{\partial}{\partial R}(-\ aR^2) + 0 + \frac{\partial}{\partial x}(2ax) = -\ 2a + 2a = 0$$



(c) Vortex lines are given by

$$\frac{dx}{\omega_x} = \frac{dR}{\omega_R} \longrightarrow \frac{dx}{2ax} = \frac{dR}{-aR}$$

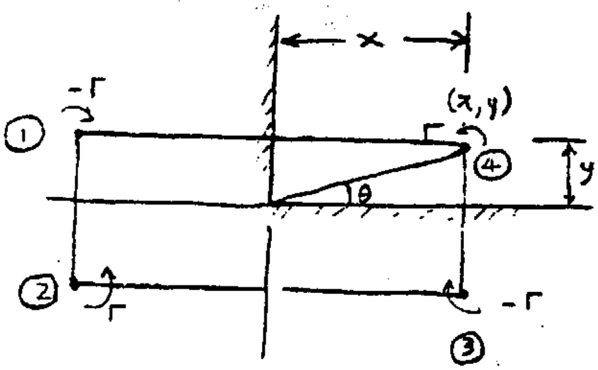


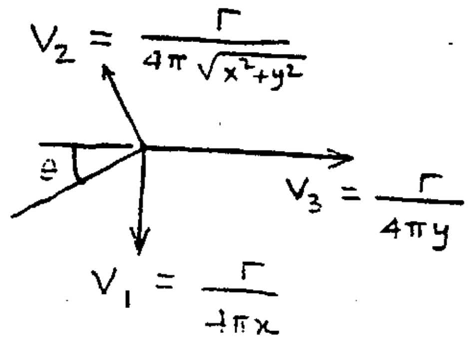
Integrating

$$\frac{1}{2}\log x = -\ \log R + \text{constant} \longrightarrow \qquad xR^2 = \text{constant}$$

Since u_φ is the only nonzero component of velocity, streamlines are circles around the x-axis. They cut the xR plane at po

Exercise 5.4





Velocity at point (x,y)

The components of the net velocity at point (x,y) due to the three image vortices are

$$u = \sum V_x = V_3 - V_2\sin\theta = \frac{\Gamma}{4\pi y} - \frac{\Gamma}{4\pi\sqrt{x^2+y^2}} \cdot \frac{y}{\sqrt{x^2+y^2}}$$

$$= \frac{\Gamma}{4\pi}\left[\frac{1}{y} - \frac{y}{x^2+y^2}\right] = \frac{\Gamma}{4\pi}\,\frac{x^2}{y(x^2+y^2)}$$

$$v = \sum V_y = -V_1 + V_2\cos\theta = -\frac{\Gamma}{4\pi x} + \frac{\Gamma}{4\pi\sqrt{x^2+y^2}}\,\frac{x}{\sqrt{x^2+y^2}}$$

$$= -\frac{\Gamma}{4\pi}\left[\frac{1}{x} - \frac{x}{x^2+y^2}\right] = -\frac{\Gamma}{4\pi}\,\frac{y^2}{x(x^2+y^2)}$$

Path lines are given by

$$dx/dt = u(t) \qquad\qquad dy/dt = v(t)$$

Therefore

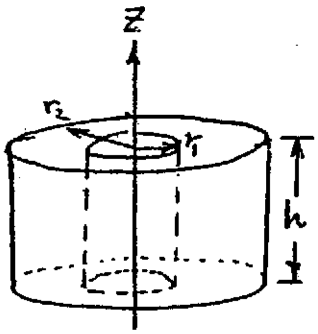
Exercise 5.7

For an irrotational vortical flow with circulation Γ, $u_\theta = \Gamma/2\pi r$. The volumetric flow rate is

$$Q = \int_{r_1}^{r_2} dr \int_0^h dz\, v_\theta = \frac{h\Gamma}{2\pi} \ln \frac{r_2}{r_1}, \qquad 1$$

$$\text{Kinetic energy} = \int_0^h dz \int_{r_1}^{r_2} dr \int_0^{2\pi} d\theta \cdot r \cdot \frac{1}{2}\rho u_\theta^2, \qquad 2$$

$$= \frac{\rho}{2}\frac{\Gamma^2}{4\pi^2} \cdot h \cdot 2\pi \int_{r_1}^{r_2} \frac{dr}{r} = \frac{\rho\Gamma^2 h}{4\pi} \ln \frac{r_2}{r_1} = \frac{1}{2}\rho\Gamma Q. \qquad 3$$



Exercise 5.8

a) $\nabla \times \mathbf{u} = \boldsymbol{\omega} = 2\boldsymbol{\Omega} = \text{const.} \qquad \int_A \boldsymbol{\omega} \cdot d\mathbf{A} = 2\Omega\pi R^2,$

b) $\int_A \boldsymbol{\omega} \cdot d\mathbf{A} = \oint_{C=\partial A} \mathbf{u} \cdot ds = \int_0^{2\pi} u_\theta \cdot R d\theta = 0,$

since $u_\theta(r = R) = 0$ by no slip.