Examples from Fluid Mechanics Potter & Foss, 1982, Great Lakes Press, MI

Example 5.1

Use the vorticity transport equation (5.14) and explain the presence of the secondary flow downstream of a bend in a river or creek. Can the typical eroded outer bank and the deposition of stones on the inner bank be explained by such a secondary flow? The velocity profile prior to the bend is as shown in Fig. E5.1. It is known that the flow speeds up on the inside of the bend, as shown at section 2.

Solution. The vorticity at the start of the bend is in the y-direction near the bottom ($\omega = \eta \hat{j}$) and in the z-direction on the inner side wall ($\omega = \zeta k$), assuming the river to form a rectangular cross-section. The flow on the inside of the bend accelerates (a fact which will be discussed in Chapter 8) resulting, at section 2, in a non-zero $\partial u/\partial y$ for the bottom flow and a non-zero $\partial u/\partial z$ for the side wall flow. Note that $\partial u/\partial y = 0$ near the bottom and $\partial u/\partial z = 0$ on the sidewall at section 1. The x-component vorticity equation (see Eq. 5.14) is

$$\frac{D\xi}{Dt} = \xi\frac{\partial u}{\partial x} + \eta\frac{\partial u}{\partial y} + \zeta\frac{\partial u}{\partial z} + \nu\nabla^2\xi$$

Thus we see that $\eta\partial u/\partial y$ from the bottom flow results in a positive $D\xi/Dt$ and hence a positive ξ at section 2. Likewise, $\zeta\partial u/\partial z$ from the sidewall results in a positive ξ. The sense of ξ is to cause a net flow from the outer bank to the

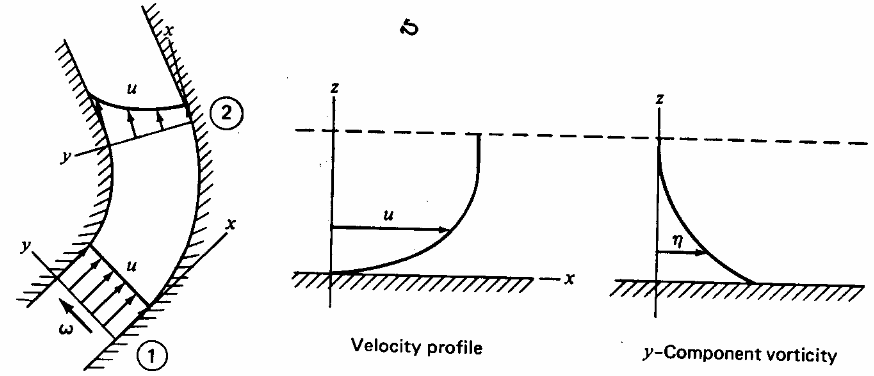


Fig. E5.1

inner bank near the bottom of the stream. (If your right thumb points in the direction of the vorticity your curled fingers will indicate the vortical motion.) This is the agent which transports stones to the inner bank. The ξ-vorticity component also causes the surface flow to move toward the outer bank after causing errosion near the stream surface. (This outward flow may be observed by dropping a leaf on the water surface upstream of the bend.)

Example 5.2

The second circulation-producing term in Eq. 5.21, $-\oint_C (dp/\rho)$, is nonzero if the density is a variable and is not related uniquely to the pressure, that is, if there is a nonbarotropic relationship between ρ and p. [It will be left as an

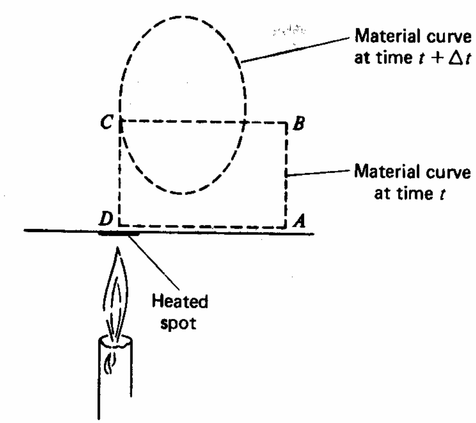


Fig. E5.2

exercise for the student to show that $\oint_C (dp/\rho) = 0$ for a barotropic flow, i.e., $\rho = \rho(p)$. Note that $\rho =$ constant satisfies this condition, as do isentropic and isothermal flows.] A nonbarotropic circulation-producing condition is the localized heating of a fluid such as would occur above a stove burner, a light bulb, or a fire. Such a situation is demonstrated in Fig. E5.2. Explain how the circulation is produced.

Solution. A closed contour of fluid particles (a material curve as shown on the figure), will experience a change in its circulation as a result of the localized heating effect. The density of the fluid above the heated spot will decrease in the region where the fluid has been heated; hence the hydrostatic-pressure variation, from some distance above the heated spot down to it, would be less than the hydrostatic-pressure change from the same initial height to the level of the spot. As a result of this, there is a pressure gradient from D to A ($p_D < p_A$) and hence a flow from A to D and from D to C. Hence, the fluid of the material curve will be carried upward. Since the pressure variation from C to D is nonbarotropic (the upward flow from D to C implies that $\partial p/\partial z > \rho g$ along CD) there will be a net contribution to $D\Gamma/Dt$ from this term. Consequently, a circulation around the material curve will be developed as the fluid of the curve rises.

Example 5.3

A relatively large boundary layer exists at the inlet to a contraction. It is observed that the boundary layer at the exit of the contraction is smaller than at the inlet (see Fig. E5.3); explain this observation.

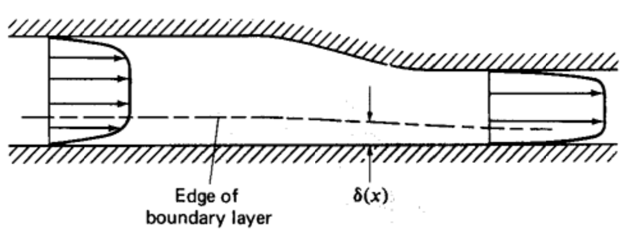
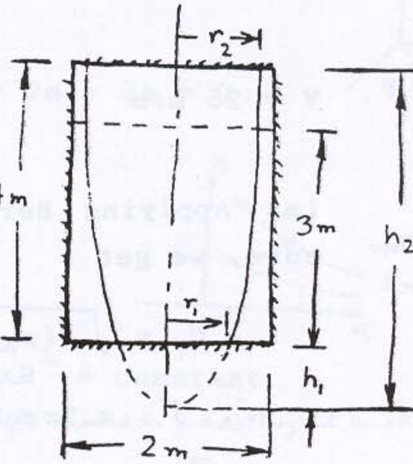


Fig. E5.3

Solution. Consider a region in space as shown on the sketch. The edge of the boundary layer is taken as the location where the outward diffusion of vorticity is negligible. For the accelerating flow, the free-stream velocity has a component directed toward the lower surface; consequently, the edge of the boundary layer will be located at the y-location where the convective transport bringing non-vortical fluid into a spatial region from above is balanced by the outward diffusion from below. For this example the boundary layer decreases in thickness with the x-location since the convective transport exceeds the viscous diffusion in the contracting region.



Exercise 5.1

Angular velocity ω = 40 rad/s

$h = \omega^2 r^2/2g$

$h_1 = \omega^2 r_1^2/2g = (40)^2 r_1^2/2(9.81) = 81.5 r_1^2$ (1)

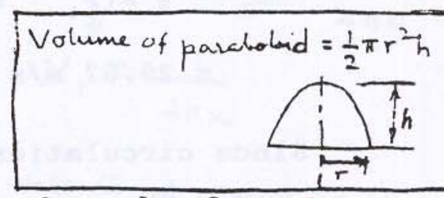
$h_2 = 4 + h_1 = \omega^2 r_2^2/2g = 81.5 r_2^2$ (2)

Subtracting (2) - (1), we get

$$4 = 81.5(r_2^2 - r_1^2) \longrightarrow \qquad r_2^2 - r_1^2 = 0.0491 \text{ m}^2 \qquad (3)$$

Now volume of air within the tank does not change, so

$$\frac{\pi}{4}(2)^2(4 - 3) = \frac{1}{2}\pi r_2^2 h_2 - \frac{1}{2}\pi r_1^2 h_1$$



or
$$2 = h_2 r_2^2 - h_1 r_1^2$$
$$= 81.5(r_2^4 - r_1^4) = 81.5(r_2^2 + r_1^2)(r_2^2 - r_1^2)$$

where we have used (1) and (2) to replace h_1 and h_2. Using (3), the above becomes

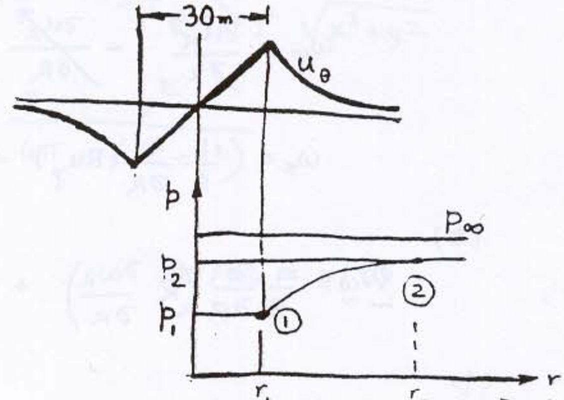
$$r_2^2 + r_1^2 = 2/(81.5)(r_2^2 - r_1^2) = 2/(81.5)(0.0491) = 0.5 \qquad (4)$$

Subtracting (4) - (3), we get $2r_1^2 = 0.5 - 0.0491 = 0.45$. Thus

$$r_1 = 0.475 \text{ m}$$

so that
$$\text{Area uncovered} = \pi r_1^2 = 0.71 \text{ m}^2$$

Exercise 5.2



$p_1 = -2000$ N/m^2

$p_2 = -500$ N/m^2

$\rho = 1.18$ kg/m^3 at 25°C

$V = 25$ m/s

(a) Applying Bernoulli equation between infinity and edge of core, we get

$$U_1 = \sqrt{2(p_\infty - p_1)/\rho} = \sqrt{2(2000)/1.18} = 58.2 \text{ m/s}$$

$$\therefore \quad \Gamma = 2\pi r_1 U_1 = 2\pi(15)(58.2) = 5485 \text{ m}^2/\text{s}$$

(b) Apply Bernoulli equation between points 2 and 1:

$$p_1 + \tfrac{1}{2}\rho U_1^2 = p_2 + \tfrac{1}{2}\rho U_2^2$$

$$U_2 = \sqrt{2(p_1 - p_2)/\rho + U_1^2} = \sqrt{2(-2000 + 500)/1.18 + 58.2^2}$$

$$= 29.07 \text{ m/s}$$

Since circulation outside core is constant, $r_1 U_1 = r_2 U_2$. So

$$r_2 = r_1 U_1 / U_2 = 30.0 \text{ m}$$

$$\text{Time required} = (r_2 - r_1)/V = (30 - 15)/25 = 0.6 \text{ s}$$

Exercise 5.3

Given

$$u_R = 0$$
$$u_\varphi = aRx$$
$$u_x = 0$$

(a) From Appendix B the vorticity components are

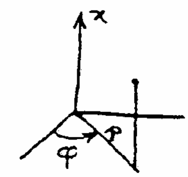
$$\omega_R = \frac{1}{R}\cancel{\frac{\partial u_x}{\partial \varphi}} - \frac{\partial u_\varphi}{\partial x} = -aR$$

$$\omega_\varphi = \cancel{\frac{\partial u_R}{\partial x}} - \cancel{\frac{\partial u_x}{\partial R}} = 0$$

$$\omega_x = \frac{1}{R}\frac{\partial}{\partial R}(Ru_\varphi) - \frac{1}{R}\cancel{\frac{\partial u_R}{\partial \varphi}} = \frac{1}{R}\frac{\partial}{\partial R}(aR^2x) = 2ax$$

(b)

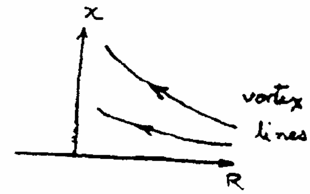
$$\nabla \cdot \omega = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \omega_R}{\partial R}\right) + \frac{1}{R}\frac{\partial \omega_\varphi}{\partial \varphi} + \frac{\partial \omega_x}{\partial x}$$



$$= \frac{1}{R}\frac{\partial}{\partial R}(-aR^2) + 0 + \frac{\partial}{\partial x}(2ax) = -2a + 2a = 0$$

(c) Vortex lines are given by

$$\frac{dx}{\omega_x} = \frac{dR}{\omega_R} \longrightarrow \frac{dx}{2ax} = \frac{dR}{-aR}$$

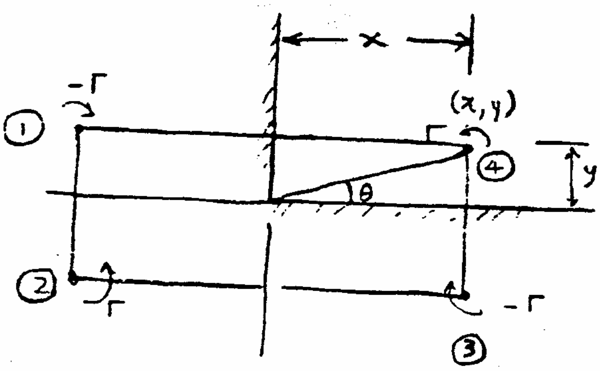


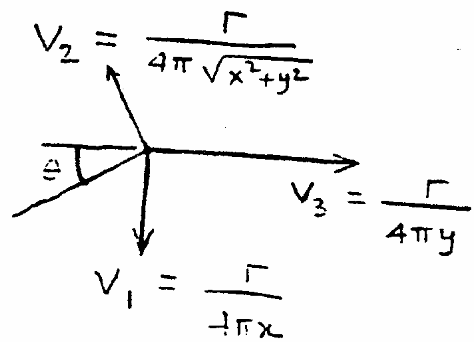
Integrating

$$\frac{1}{2}\log x = -\log R + \text{constant} \longrightarrow \quad xR^2 = \text{constant}$$

Since u_φ is the only nonzero component of velocity, streamlines are circles around the x-axis. They cut the xR plane at po

Exercise 5.4





Velocity at point (x,y)

The components of the net velocity at point (x,y) due to the three image vortices are

$$u = \sum V_x = V_3 - V_2 \sin\theta = \frac{\Gamma}{4\pi y} - \frac{\Gamma}{4\pi\sqrt{x^2+y^2}} \cdot \frac{y}{\sqrt{x^2+y^2}}$$

$$= \frac{\Gamma}{4\pi}\left[\frac{1}{y} - \frac{y}{x^2+y^2}\right] = \frac{\Gamma}{4\pi}\frac{x^2}{y(x^2+y^2)}.$$

$$v = \sum V_y = -V_1 + V_2\cos\theta = -\frac{\Gamma}{4\pi x} + \frac{\Gamma}{4\pi\sqrt{x^2+y^2}}\frac{x}{\sqrt{x^2+y^2}}$$

$$= -\frac{\Gamma}{4\pi}\left[\frac{1}{x} - \frac{x}{x^2+y^2}\right] = -\frac{\Gamma}{4\pi}\frac{y^2}{x(x^2+y^2)}$$

Path lines are given by

$dx/dt = u(t)$ $\qquad\qquad$ $dy/dt = v(t)$

Therefore

$$\frac{dx}{dt} = \frac{\Gamma}{4\pi}\left[\frac{x^2}{y(x^2+y^2)}\right]$$

$$\frac{dy}{dt} = -\frac{\Gamma}{4\pi}\left[\frac{y^2}{x(x^2+y^2)}\right]$$

Dividing we get

$$dy/dx = -y^3/x^3 \longrightarrow dy/y^3 = -dx/x^3$$

Integration gives

$$1/x^2 + 1/y^2 = F(t)$$

Initial condition gives $1/x_o^2 + 1/y_o^2 = F(t)$, where (x_o, y_o) are the initial coordinates of the vortex. Thus F(t) has to be a constant.

<u>Exercise 5.5</u>

Equation of motion in rotating coordinates is

$$\frac{D\underset{\sim}{u}}{Dt} = -\frac{1}{\rho}\underset{\sim}{\nabla} p + \underset{\sim}{g} - 2\underset{\sim}{\Omega} \times \underset{\sim}{u}$$

In Section 5.4, we took the dot product of each term with $d\underset{\sim}{x}$. Under barotropic and inviscid conditions, and in the presence of conservative body forces, we showed that

$$\frac{D}{Dt}(\Gamma) = 0$$

The extra term here is the Coriolis force. Taking its dot product with element $d\underset{\sim}{x}$ parallel to a circuit C, we get

$$-(2\underset{\sim}{\Omega} \times \underset{\sim}{u})\cdot d\underset{\sim}{x} = -2(\underset{\sim}{\Omega} \times \underset{\sim}{u})_i\, dx_i = -2\varepsilon_{ijk}\Omega_j\, u_k dx_i = -2\Omega_j \varepsilon_{ijk} u_k dx_i$$

$$= 2\Omega_j \varepsilon_{jik}\, dx_i u_k = 2\Omega_j (d\underset{\sim}{x} \times \underset{\sim}{u})_j = -2\Omega\cdot(\underset{\sim}{u} \times d\underset{\sim}{x})$$

$$= -2\underset{\sim}{\Omega}\cdot\underset{\sim}{n}\, u_\perp\, dx \qquad (1)$$

where $\underset{\sim}{n}$ is the unit vector perpendicular to the plane of $\underset{\sim}{u}$ and $d\underset{\sim}{x}$, and $u_\perp$ is the component of $\underset{\sim}{u}$ perpendicular to $d\underset{\sim}{x}$. In an